UNIVERSITÀ DEGLI STUDI DI TRIESTE

Valuation of GLWB variable annuities with accumulation phase and LTC option

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Agenda

- The contract structure
- The valuation problem
- Dynamic programming & Bang Bang analysis
- Numerical examples

The contract structure

Consider a GLWB VA contract that includes an initial accumulation phase and an LTC option.

- single-premium policy

  0 time of contract inception

  x policyholder's age at the inception

  P single premium entirely invested in a well-diversified fund.
- right to make periodical withdrawals at some specified dates for the whole life, even if the account value is reduced to zero, and to receive LTC benefits in the event of disability.
- additional purchases of funds, up to a constant fraction $0 < \pi < 1$ of the benefit base are allowed in the accumulation phase as well as dynamic withdrawals in the income phase, and complete surrender rights throughout the whole life of the contract.
- upon the policyholder death the remaining policy account is paid to the beneficiary as a death benefit.

The contract structure

S_t: market price at time t of each unit of the fund, that drives the return on the investment portfolio built up with the policyholder's payment.

W_t: value at time t of such portfolio, called policy account

φ: insurance fees to finance the cost of the GLWB rider + LTC option by periodical proportional deductions from W_t

A_t: the benefit base A_t, which is initially set equal to the single premium.

We assume that:
1) withdrawals are allowed on a predetermined set of equidistant dates and we take the distance between two consecutive dates as unit of measurement of time;
2) the death benefit is paid to the beneficiary on the next upcoming withdrawal date.

The contract structure

Let τ be the time of death of the policyholder, so that withdrawals are allowed only at times $j = 1, 2, \ldots,$ provided that $\tau > j$. Moreover, let λ be the time of permanent disability with the convention that if $\lambda \geq \tau$ the policyholder never becomes disabled. The variable

$$z_j = \mathbf{1}_{\{\lambda > j\}} = \begin{cases} 1 & \text{if} \quad \lambda > j \\ 0 & \text{if} \quad \lambda \leq j \end{cases},$$

which defines the healthy status of the policyholder at time j.
According to the policyholder's health status at time $j = 1, \ldots, T, T+1, \ldots,$ we consider a different withdrawal rate: let $g_j^{(i)} = g^{(i)}$ be the withdrawal rate of the policyholder which is disabled (i) at time j, while let

$$g_j^{(h)} = \begin{cases} 0 & \text{if} \quad j \leq T \\ g^{(h)} & \text{if} \quad j > T \end{cases},$$

be the withdrawal rate of the healthy (h) policyholder, with $g^{(i)} \neq g^{(h)}$.
The guaranteed amount that can be withdrawn at time $j = 1, \cdots, T$ is equal to $g^{(i)} A_j$, while for $j = T+1, \ldots,$ it is equal to $g^{(m)} A_j$, $m = h, i$

The contract structure

The return on the reference fund over the interval $[j-1, j]$ is

$$R_j = (S_j/S_{j-1}) - 1, \; j = 1, 2, \dots \; .$$

– y_j the actual withdrawal made by the policyholder at time j. It is admissible if it belongs to the set of admissible withdrawal strategies $Y = (Y_j)_{j \in \mathbb{N}^+}$ where

$$Y_j = Y_j^{(h)} \mathbf{1}_{\{\lambda > j\}} + Y_j^{(i)} \mathbf{1}_{\{\lambda \leq j\}},$$

$$Y_j^{(m)} = \begin{cases} [-\pi A_j, W_j], & m = h; \; j = 1, \dots, T \\ [-\pi A_j, \max\{g^{(i)} A_j, W_j\}], & m = i; \; j = 1, \dots, T \\ [0, \max\{g^{(m)} A_j, W_j\}], & m = h, i; \; j = T+1, \dots \end{cases} . \qquad (1)$$

– If $y_j = 0$ then A_j is increased according to the roll-up rate $b_j^{(m)} \in (0,1)$, $m = h, i$, and according to any possible additional purchase allowed in the accumulation phase only; while, if $y_j > g^{(m)} A_j$, it is proportionally reduced according to the so called 'pro-rata' adjustment rule.

The contract structure

To describe the evolution of the benefit base we introduce the following function

$$a_{j+1}^{(m)}(W_j, A_j, y_j) = \begin{cases} A_j\left(1 + b_j^{(m)}\right) - y_j & \text{if} \quad y_j \leq 0, \\ A_j & \text{if} \quad 0 < y_j \leq g^{(m)} A_j, \\ A_j \dfrac{W_j - y_j}{W_j - g_j^{(m)} A_j} & \text{if} \quad g_j^{(m)} A_j < y_j \leq W_j \end{cases},$$

$m = h, i,\ j = 1, 2, \ldots,$ from which we get the evolution of the benefit base:

$$A_{j+1} = a_{j+1}^{(h)}(y_j, W_j, A_j)\mathbf{1}_{\{\lambda > j\}} + a_{j+1}^{(i)}(y_j, W_j, A_j)\mathbf{1}_{\{\lambda \leq j\}},$$

with $A_1 = P$. If $y_j > g_j^{(m)} A_j$, there is also a proportional penalization on the surplus according to a penalty rate $k_j^{(m)} \in [0, 1)$.[1] Therefore, the net amount (cash-flow) received at time j is given by

$$C_j^{(m)}(y_j, A_j) = y_j - k_j^{(m)} \max\{y_j - g_j^{(m)} A_j, 0\}, \quad m = h, i;\ j = 1, 2, \ldots\ ;\ y_j \in Y_j.$$

[1] Note that this penalization applies also in the case there is no guaranteed amount, namely in the accumulation phase for the healthy policyholder if she withdraws something bigger than zero.

The contract structure

The policy account value evolves according to the following equation:

$$W_{j+1} = w(W_j, R_{j+1}, y_j) = \max\{W_j - y_j, 0\}(1 + R_{j+1})(1 - \varphi), \quad j = 0, 1, \ldots,$$

where $\varphi \in (0,1)$, $W_0 = P$ and $y_0 = 0$.

Note that 0 is an absorbent barrier for W because, once it becomes null, it remains so for ever. The contract, however, continues while $A_t > 0$ (and the insured is still alive). Finally, in case of death in the time interval $(j-1, j]$, the death benefit, paid at time j, is

$$B_j = W_j, \quad j - 1 < \tau \leq j\ , \quad j = 1, 2, \ldots\ .$$

In case of surrender at time j, i.e., when $y_j = W_j > g_j^{(m)} A_j$, the contract is automatically closed for all $t > j$, hence no further withdrawals are admitted, nor a death benefit will be paid.

The valuation problem

- We assume to act in a frictionless and arbitrage-free market and let $(\Omega, \mathcal{F}, Q)$ be a complete probability space equipped with a complete and right continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, where Q is the risk-neutral probability measure selected by the insurer among the infinitely many equivalent martingale measures existing in incomplete markets.
- independence between biometric and financial risks, as well as deterministic transition and death probabilities.
- Given r the instantaneous interest rate which we assume to be deterministic and constant, we model the reference portfolio value S_t as an exponential Lévy process:

$$S_t = S_0 \mathrm{e}^{(r+d)t + X_t},$$

where $(X_t)_{t\geq 0}$, with $X_0 = 0$, is a $(\mathbb{F}, Q)$-Lévy process, and d represents an adjustment so that $S_t\,\mathrm{e}^{-rt}$ is a $(\mathbb{F}, Q)$-martingale.

The valuation problem

Let

$$\left.\begin{aligned} p_j^{h,h} &= Q(\tau > j+1,\ z_{j+1} = 1 \,|\, z_j = 1, \tau > j), \\ p_j^{h,i} &= Q(\tau > j+1,\ z_{j+1} = 0 \,|\, z_j = 1, \tau > j), \end{aligned}\right. \qquad j = 0, 1, \ldots,$$

be the one-year transition probabilities which represent, respectively, the probability to be alive and healthy at age $x + j + 1$, and to be alive but disabled at age $x + j + 1$, both conditional on survival and to being healthy at age $x + j$. Moreover, let

$$p_j^{i,i} = Q(\tau > j+1,\ z_{j+1} = 0 \,|\, z_j = 0, \tau > j), \qquad j = 1, 2, \ldots,$$

be the probability to be alive and disabled at age $x + j + 1$ conditional on survival and to being disabled at age $x + j$. Note that $p_j^{i,h} = 0$ for the hypothesis of permanent disability. Concerning the death probabilities, let

$$\begin{aligned} q_j^h &= Q(\tau \leq j+1 \,|\, \tau > j, z_j = 1), \qquad j = 0, 1, \ldots, \\ q_j^i &= Q(\tau \leq j+1 \,|\, \tau > j, z_j = 0), \qquad j = 1, 2, \ldots, \end{aligned}$$

be the probabilities of dying before age $x + j + 1$ conditional on survival and to being healthy or disabled, respectively, at age $x + j$.

The valuation problem

The initial value of the GLWB variable annuity with the LTC option is the solution of the following problem:

$$V_0 = \sup_{y \in Y} \mathbb{E}^Q \left[\sum_{j=1}^{n} \mathrm{e}^{-rj} \left(\mathbf{1}_{\{\tau \wedge \lambda > j\}} C_j^{(h)}(y_j, A_j) + \mathbf{1}_{\{\lambda \leq j < \tau\}} C_j^{(i)}(y_j, A_j) + \mathbf{1}_{\{j-1 < \tau \leq j\}} W_j \right) + \mathrm{e}^{-r(n+1)} \mathbf{1}_{\{\tau > n\}} W_{n+1} \right],$$

where $n = \omega - x$, with ω denoting the maximum attainable age[2] for the policyholder, beyond which her survival probability is zero.

- For simplicity, we assume the same maximum age ω for both healthy and disabled policyholders.

[2] Our unit of measurement of time is the common distance between two consecutive withdrawal dates, so that also x and ω are expressed according to this measure.

Dynamic Programming

Let $V_j(W_j, A_j, z_j)$ be the contract value at time j (before the periodic withdrawal); $v_j(W_j, A_j, z_j)$ the contract value at the same time when the policyholder is alive.

$$V_j(W_j, A_j, z_j) = \mathbf{1}_{\{\tau > j\}} v_j(W_j, A_j, z_j) \qquad V_0 = V_0(P, P, 1) = v_0(P, P, 1).$$

We take $n + 1$ as the starting point of our backward dynamic algorithm, and define the following terminal condition:

$$v_{n+1}(W_{n+1}, A_{n+1}, z_{n+1}) \equiv 0.$$

Then, we proceed backward and for $j = n, n-1, \ldots, 1$, we define the Bellman recursive equation of the problem as:

$$\begin{aligned} v_j(W_j, A_j, z_j) = \sup_{y_j \in Y_j} \Bigg(& \mathbf{1}_{\{\lambda > j\}} \, C_j^{(h)}(y_j, A_j) + \mathbf{1}_{\{\lambda \leq j\}} \, C_j^{(i)}(y_j, A_j) \\ & + \mathbb{E}^Q \left[\mathbf{1}_{\{\tau \leq j+1\}} w(W_j, R_{j+1}, y_j) \mathrm{e}^{-r} | W_j, A_j, z_j, \tau > j \right] \\ & + \mathbb{E}^Q \Big[\mathbf{1}_{\{\tau > j+1\}} v_{j+1} \big(w(W_j, R_{j+1}, y_j), a_{j+1}^{(h)}(W_j, A_j, y_j) \, \mathbf{1}_{\{\lambda > j\}} \\ & \quad + a_{j+1}^{(i)}(W_j, A_j, y_j) \, \mathbf{1}_{\{\lambda \leq j\}}, z_{j+1} \big) \mathrm{e}^{-r} | W_j, A_j, z_j, \tau > j \Big] \Bigg) \end{aligned}$$

Dynamic Programming

$$
\begin{aligned}
v_j(W_j, A_j, z_j) = \sup_{y_j \in Y_j} \Bigg(& \mathbf{1}_{\{\lambda > j\}} \; C_j^{(h)}(y_j, A_j) \; + \; \mathbf{1}_{\{\lambda \le j\}} \; C_j^{(i)}(y_j, A_j) \\
& + \max\{W_j - y_j, 0\}(1-\varphi) Q(\tau \le j+1 \,|\, z_j, \tau > j) \\
& + \mathbb{E}^Q \Big[v_{j+1} \Big(w(W_j, R_{j+1}, y_j), a_{j+1}^{(h)}(W_j, A_j, y_j)\, \mathbf{1}_{\{\lambda > j\}} + a_{j+1}^{(i)}(W_j, A_j, y_j)\, \mathbf{1}_{\{\lambda \le j\}}, 1 \Big) \, \mathrm{e}^{-r} \\
& \qquad \,|\, W_j, A_j, z_j \Big] \, Q(\tau > j+1, \, z_{j+1} = 1 \,|\, z_j, \tau > j) \\
& + \mathbb{E}^Q \Big[v_{j+1} \Big(w(W_j, R_{j+1}, y_j), a_{j+1}^{(h)}(W_j, A_j, y_j)\, \mathbf{1}_{\{\lambda > j\}} + a_{j+1}^{(i)}(W_j, A_j, y_j)\, \mathbf{1}_{\{\lambda \le j\}}, 0 \Big) \, \mathrm{e}^{-r} \\
& \qquad \,|\, W_j, A_j, z_j \Big] \, Q(\tau > j+1, \, z_{j+1} = 0 \,|\, z_j, \tau > j) \Bigg)
\end{aligned}
$$

Finally, the initial contract value is:

$$
\begin{aligned}
v_0(P, P, 1) = q_0^h P(1-\varphi) + & \mathbb{E}^Q \Big[v_1(P(1+R_1)(1-\varphi), P, 1)\, e^{-r} \Big] p_0^{h,h} \\
+ & \mathbb{E}^Q \Big[v_1(P(1+R_1)(1-\varphi), P, 0)\, e^{-r} \Big] p_0^{h,i}.
\end{aligned}
$$

Bang Bang analysis

At each time step $j = n, n-1, \ldots 1$, the Bellman equation requires to solve:

- a real-valued optimization problem on $Y_j^{(m)}$ (1) whose expression depends both on the health status of the policyholder and the contract phase.

Since the computational effort could be substantial, a property that drastically reduces this effort is the bang-bang condition which is satisfied for our problem.

Proposition

The optimal withdrawal strategy is $y_j \in \{-\pi A_j, 0, W_j\}$ for $j = 1, \ldots, T$ (i.e., in the accumulation phase) if the policyholder is healthy, and $y_j \in \{-\pi A_j, 0, g^{(i)} A_j, W_j\}$ if disabled. For $j = T+1, \ldots n$ (i.e., in the income phase) it is $y_j \in \{0, g^{(m)} A_j, W_j\}$ for $m = h, i$.

Proof: by backward induction, starting from the case $j = n$. Then, assuming the proposition holds at step $j+1$, for $j = n-1, \ldots, T+1$, one proves that it holds for $j = T+1$ which serves as starting condition for the iterative step 2. Therefore, by assuming the proposition holds for $j = T, T-1, \ldots, 1$, one gets that it holds for $j = 1$.

Bang Bang analysis

In the accumulation phase, if the PH is healthy:

- it may be convenient to purchase $y_j = -\pi A_j$, but never to purchase an amount y_j with $-\pi A_j < y_j < 0$, then the optimal strategy is $y_j = 0$.
- it is possible to withdraw $y_j > 0$ subject to the withdrawal penalty $k_j y_j$ as well as to a second penalization due to the reduction of the benefit base according to the pro-rata rule. Then it is convenient that $y_j = W_j$.

For the disabled PH, the same arguments apply to $y_j = -\pi A_j$ and $y_j = 0$.

- It is never convenient to withdrawal $0 < y_j < g^{(i)} A_j$ as the policyholder loses the roll-up incentive without taking full advantage of the guarantee.
- if $W_j > g^{(i)} A_j$, it is possible to withdraw $y_j > g^{(i)} A_j$ subject to the withdrawal penalty $k_j(y_j - g^{(i)} A_j)$. Moreover, if $g^{(i)} A_j < y_j < W_j$, there is a second penalization due to the pro-rata adjustment rule. Therefore, it would be more convenient that $y_j = W_j$.

In the income phase, we come to similar conclusions as above and thus the set of optimal withdrawals for the healthy policyholder is $\{0, g^{(h)} A_j, W_j\}$, while for the disabled policyholder is $\{0, g^{(i)} A_j, W_j\}$.

Numerical examples

- We fix P =100, annual withdrawals and an insured aged $x = 50$ at contract inception, marking the beginning of the accumulation phase, which lasts 15 years. Consequently, the insured is aged $x = 65$ when the income phase begins.
- We adopt the projected transition and death probabilities stemming from the study of Baione et al. 2016 that exploits health-related data provided by INPS to fit a three-state (healthy, disabled, dead), continuous time Markov model. As a result, we take $\omega = 120$, and we consider the cohort of the Italian population aged $x = 50$ in 2013, following that $n = 70$.
- VG model whose characteristic function is

$$\Phi_t(u) = \exp\left(-\frac{t}{v}\ln\left(1 - iu\mu v + \frac{1}{2}\sigma^2 u^2 v\right)\right),$$

  with $\mu \in \mathbb{R}$, σ, $v > 0$.
- $\sigma = 0.2$, $v = 0.85$, and $\mu = 0$, which were calibrated using S&P 500 market data[3]
- $r = 3.5\%$ consistent with the term structure of risk-free interest rates in the U.S. market for very long maturities.

[3]

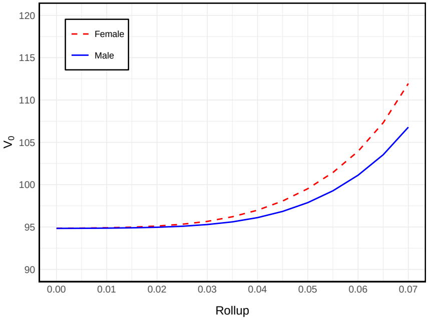
Numerical examples

The contract parameters in the basic case are:

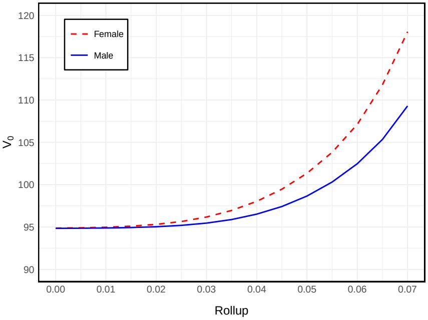
Contract parameter	Notation	Value
Bonus rate	$b_j^{(h)} = b_j^{(i)}$	5%
Penalty rate	$k_j^{(h)} = k_j^{(i)}$	3%
Withdrawal rate (healthy)	$g^{(h)}$	3%
Withdrawal rate (disabled)	$g^{(i)} = 2\, g^{(h)}$	6%
Fee rate	φ	2.34%
Withdrawal dates		yearly

- Note that he fair fee rate for a male should be equal to 1.96% and the one for a female should be equal to 2.72%. We have chosen a basic level for $\varphi = 2.34\%$ as the average of these two values. This implies that the contract is overpriced (i.e., $V_0 < P$) for male policyholders while it is underpriced (i.e., $V_0 > P$) for female policyholders.

Numerical examples



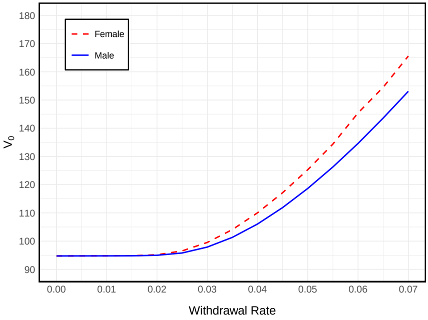
(a) V_0 without LTC option



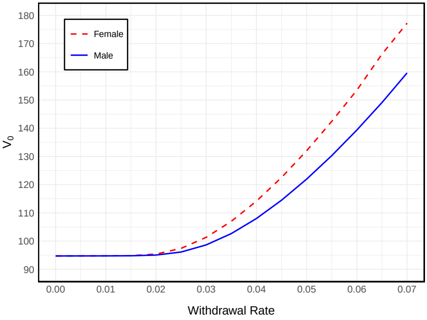
(b) V_0 with LTC option

Figure: Initial contract value without and with the LTC option for different roll-up rates.

Numerical examples



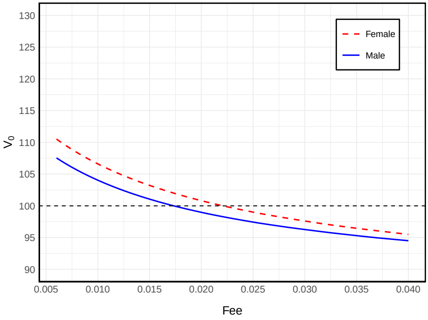
(a) V_0 without LTC option



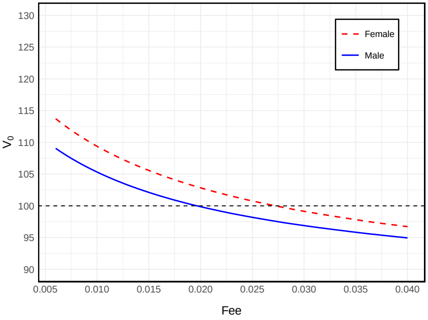
(b) V_0 with LTC option

Figure: Initial contract value without and with the LTC option for different withdrawal rates.

Numerical examples



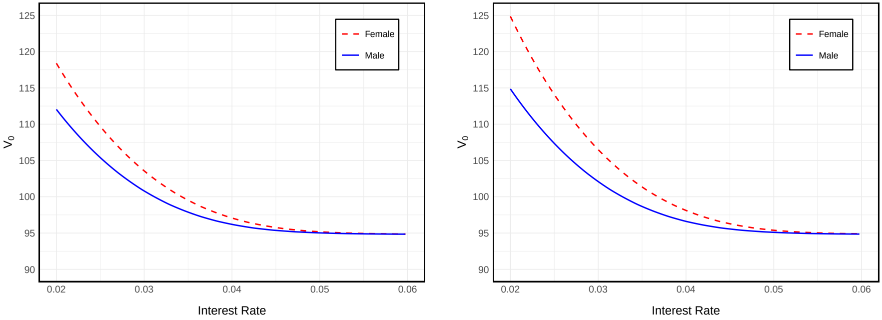
(a) V_0 without LTC option



(b) V_0 with LTC option

Figure: Initial contract value without and with the LTC option for different fee rates.

Numerical examples



(a) V_0 without LTC option (b) V_0 with LTC option

Figure: Initial contract value without and with the LTC option for different interest rates.

Numerical examples

Table: Fair fee for different roll-up rates b

Case	$b = 0.0425\,(-15\%)$	$b = 0.0475\,(-5\%)$	$b = 0.05$	$b = 0.0525\,(+5\%)$	$b = 0.0575\,(+15\%)$
			Fair fee		
Male	**0.0143** (-27.04%)	**0.0176** (-10.20%)	**0.0196**	**0.0218** $(+11.22\%)$	**0.0270** $(+37.75\%)$
Female	**0.0197** (-27.57%)	**0.0244** (-10.29%)	**0.0272**	**0.0304** $(+11.76\%)$	**0.0382** $(+40.44\%)$

Table: Fair fee for different withdrawal rates g

Case	$g = 0.0255\,(-15\%)$	$g = 0.0285\,(-5\%)$	$g = 0.03$	$g = 0.0315\,(+5\%)$	$g = 0.0345\,(+15\%)$
			Fair fee		
Male	**0.0123** (-37.24%)	**0.0169**(-13.77%)	**0.0196**	**0.0226** $(+15.30\%)$	**0.0296** $(+51.02\%)$
Female	**0.0170** (-37.50%)	**0.0234** (-13.97%)	**0.0272**	**0.0315** $(+15.80\%)$	**0.0421** $(+54.78\%)$

Thank you!

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