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Skew Brownian motion discretization: A lattice approach for financial and actuarial applications

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Introduction

Outline

Introduction: motivations and aim.

Model features:

- novelty: It represents one of the first models able to price American-type derivatives under a skew Brownian motion (sBm);
- flexibility: It is useful not only in financial but also in actuarial applications where such claims are embedded in several structured insurance policies, e.g., equity-linked policies with surrender options;
- simplicity and efficiency: Simple to implement and efficient for valuing contingent claims in terms of accuracy of the results with respect to the benchmark and speed of execution with reference to the computational time;
- bivariate construction: It arises from the combination of two lattices discretizing the Brownian motion (Bm) and the reflecting Brownian motion (rBm), respectively, appearing in the sBm process.

Numerical results for model validation

Concluding remarks and hints to future works.

Motivations

Motivation 1. Cont and Tankov (2004) and several empirical studies therein referenced highlight how financial returns are characterized by stochastic volatility and present fatter tails with respect to the standard normal model.

Motivation 2. The **Black and Scholes (1973)** (BS) model does not describe accurately such dynamics and produces biases in option prices, underpricing deep in-the-money options and overpricing deep out-of-the-money options.

Motivation 3. The volatility smile (according to which stock price volatility depends upon the option strike price and time-to-maturity) implies the existence of a term structure of implied volatilities that cannot be replicated by the BS model since it is based on a constant volatility parameter.

Motivation 4. The biases introduced in option pricing by the BS model have led to the exploration of alternative models that more accurately capture the dynamics of the underlying stock price. Among others, a sBm characterized by a skew-normal distribution is one of the candidate useful to the scope.

Aim

Main Aim. The proposal of a novel lattice approach to discretize a sBm.

The proposed lattice-based approach starts by discretizing independently both the Bm and the rBm appearing in the sBm by means of two distinct individual **Cox and Rubinstein (1985)** binomial recombining trees characterized by constant transition probabilities equal to $\frac{1}{2}$.

Such two trees are combined in order to establish a bivariate lattice with each node presenting four branches that are useful to capture all the possible individual tree movements.

Due to the independence between the Bm and the rBm, the probability associated with each branch of the bivariate lattice is obtained by simply multiplying the marginal probabilities characterizing the corresponding movements in each individual tree. It means that all the branches have the same occurrence probability equal to $\frac{1}{4}$.

The lattice-based discretization

Framework

Working in the same settings of **Corns and Satchell (2007)** and **Zhu and He (2018)**, we hypothesize that the process describing the underlying asset price in the interval $[t, T]$ under the risk-neutral probability measure $\mathbb{Q}$ is given by

$$S_T = S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)-l(z)+\sigma(\bar{X}_T-\bar{X}_t)},$$

where t represents the current time, r is the risk-free rate in the market, σ is the process volatility, the process $\bar{X}_s$ is a **Itô and McKean (1965)** sBm characterized by a skew-normal probability density function that allows to define $\bar{X}_s$ as a linear combination of a Bm and a rBm, i.e.,

$$\bar{X}_s = \sqrt{1-\delta^2}\bar{W}_{1,s} + \delta|\bar{W}_{2,s}|, \quad \delta \in (-1, 1),$$

where $\bar{W}_1$ and $\bar{W}_2$ are independent Bms, and

$$l(z) = \log\left(\Phi\left[\frac{z+(T-t)\sigma^2\delta^2}{\sigma\delta\sqrt{T-t}}\right] + e^{-2z}\Phi\left[\frac{-z+(T-t)\sigma^2\delta^2}{\sigma\delta\sqrt{T-t}}\right]\right),$$

with $z = \sigma\delta|\bar{W}_{2,t}|$.

Lattice construction

We split the time horizon $[t, T]$ into n subintervals of equal length $\Delta t = \frac{T-t}{n}$ and define, for $i = 0, \ldots, n$, node $(i, 0)$ as the lowest node at time $t + i\Delta t$, node $(i, 1)$ as the second to the lowest, and so on up to the highest node (i, i). Hence, the lattice is rooted at node $(0, 0)$ at time t while the underlying asset prices at time $t + i\Delta t$ are described by the nodes $(i, j),\ j = 0, \ldots, i$.

We start by independently discretizing both the Bm $\bar{W}_1$ and the rBm $|\bar{W}_2|$, by means of two distinct individual binomial recombining trees.

For $x = 1, 2$, the discrete increments over a time interval of length Δt, $\Delta\bar{W}_x$, of the Brownian motion $\bar{W}_x$, may be defined according to the classical **Cox and Rubinstein (1985)** scheme as

$$\Delta\bar{W}_x = \begin{cases} \sqrt{\Delta t} & \text{with probability} \quad \frac{1}{2} \\ -\sqrt{\Delta t} & \text{with probability} \quad \frac{1}{2} \end{cases} .$$

Bm and rBm values

Denoting by $\bar{W}_x(i,j),\ j = 0,\dots,i$, the discrete value of the $\bar{W}_x$-process in correspondence with node (i,j), we can easily compute

$$\bar{W}_x(i,j) = \bar{W}_x(0,0) + (2j-i)\Delta t, \quad \text{where} \quad \bar{W}_x(0,0) = \bar{W}_{x,t}.$$

The reflecting Brownian motion value in correspondence with node (i,j) is obtained as the absolute value of $\bar{W}_2(i,j)$, i.e.,

$$|\bar{W}_2|(i,j) = |\bar{W}_2(0,0) + (2j-i)\Delta t|.$$

Under these constructions the probability associated with each possible upward or downward movement in the generated lattices is always equal to $\frac{1}{2}$ and the successors for $\bar{W}_x(i,j)$ are

$$\bar{W}_x(i+1,j+1) \quad \text{and} \quad \bar{W}_x(i+1,j), \quad \text{respectively.}$$

The bivariate model

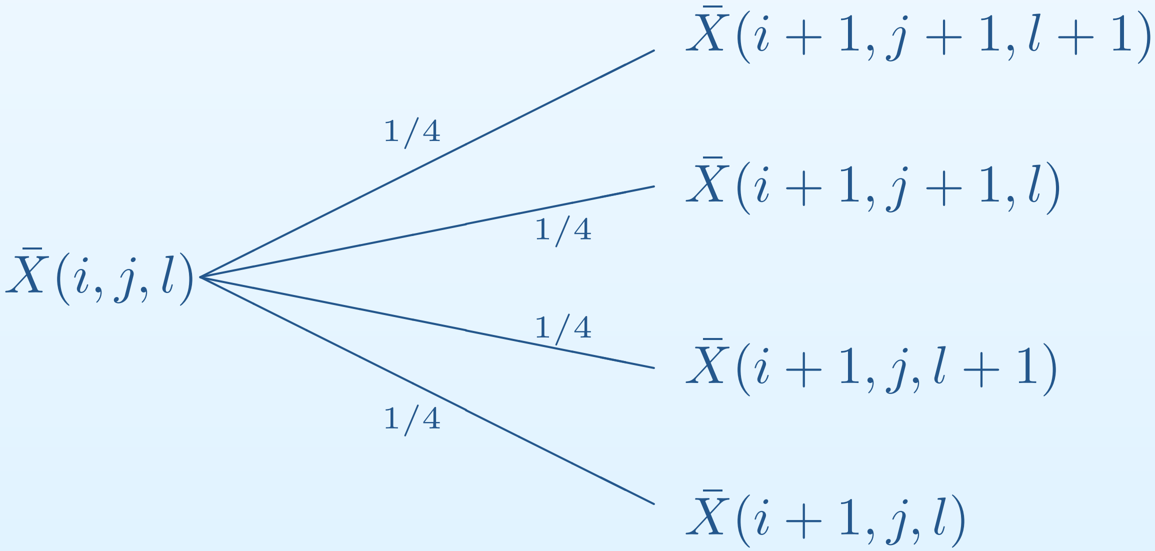
The bivariate lattice...

The lattice values discretizing $\bar{W}_1$ and $|\bar{W}_2|$ must be combined at each time slice $t + i\Delta t,\ i = 0, \ldots, n$, in order to discretize the process $\bar{X}$.

$$\Downarrow$$

The $\bar{X}$-process is discretized through a bivariate lattice (BL) in which each node presents four branches to capture all the possible individual lattice movements.

The bivariate lattice states are defined as a triplet (i, j, l) with $j, l = 0, \ldots, i$, in correspondence of which the Bm assumes value $\bar{W}_1(i, j)$ and the rBm has value $|\bar{W}_2|(i, l)$, and along each branch emanating from a generic state (i, j, l), each individual lattice may show a downward or an upward step.



...in formulae

We compute the asset value in each state (i, j, l) starting from time t where the BL is rooted at node $(0, 0, 0)$ and the observed asset price is $S(0, 0, 0) = S_t$. In particular, when moving from state $(0, 0, 0)$ to state (i, j, l), the time horizon has width equal to $i\Delta t$ and we need the corresponding discrete time version of $l(z)$, $\bar{X}(i, j, l)$ and $\bar{X}(0, 0, 0)$, to compute the asset value $S(i, j, l)$ that is consequently given by

$$S(i, j, l) = S(0, 0, 0)e^{\left(r-\frac{1}{2}\sigma^2\right)i\Delta t - l(z) + \sigma(\bar{X}(i,j,l) - \bar{X}(0,0,0))},$$

where

$$\bar{X}(i, j, l) = \sqrt{1 - \delta^2}\bar{W}_1(i, j) + \delta|\bar{W}_2|(i, l),$$

and $l(z)$ is obtained as

$$l(z) = \log\left(\Phi\left[\frac{z + i\Delta t\sigma^2\delta^2}{\sigma\delta\sqrt{i\Delta t}}\right] + e^{-2z}\Phi\left[\frac{-z + i\Delta t\sigma^2\delta^2}{\sigma\delta\sqrt{i\Delta t}}\right]\right),$$

with $z = \sigma\delta|\bar{W}_2|(0, 0) = \sigma\delta|\bar{W}_{2,t}|$.

Applications

European and American options

Suppose to consider a European call option with strike price K issued at time $t = 0$ and maturing at time T.

Labelling by $V(i, j, l),\ i = 0, \dots, n,\ j, l = 0, \dots, i$, the option value in correspondence to state (i, j, l), we start from maturity where, on the terminal states of nature (n, j, l), the option payoff is given by

$$V(n, j, l) = \max(S(n, j, l) - K, 0).$$

Proceeding backward, the option value in correspondence of state (i, j, l) is computed by discounting at the risk-free rate r the option value associated with the successors of state (i, j, l) having the same occurrence probability $\frac{1}{4}$,

$$V(i, j, l) = \frac{1}{4} e^{-r\Delta t} \left[V(i+1, j+1, l+1) + V(i+1, j+1, l) + \right.$$
$$\left. V(i+1, j, l+1) + V(i+1, j, l) \right].$$

Whenever the considered option has American-style features, we have to embed the early exercise feature in the backward formula above.

Equity-linked policy features

The second application we propose is focused on the valuation of a single premium equity-linked term policy, issued at time $t = 0$ and maturing at time T, equipped with a minimum guarantee.

Such a guarantee has the function of protecting the policyholder's investment against a bad performance of the fund. It means that now the S-dynamics describes the fund fluctuations over time and the insurer is forced to pay at least a minimum amount G.

Without loss of generality, we choose G equal to the initial investment in the fund, i.e., $G = S_0$, but other forms for G may be easily managed in the proposed bivariate model.

At maturity T, the insurer refunds the policyholder with the policy payoff that is equal to $\max[S(T), G]$.

To compute the policy fair value, i.e., the fair single premium paid by the policyholder, we can apply the **Brennan and Schwartz (1976)** decomposition that allows us to to write the policy payoff at maturity as a function of a financial option payoff.

Equity-linked policy evaluation

In particular, we can obtain a call-decomposition as

$$G + \max[S(T) - G, 0],$$

and we compute the value of the call option by defining, at first, $V(n, j, l) = \max(S(n, j, l) - G, 0)$, and then applying the recursive formula to compute $V(0, 0, 0)$ at inception. The policy fair value at inception is computed by summing up the option price and the quantity Ge^{-rT}.

If we embed in the contract a surrender option, we define the surrender amount in each state (i, j, l) as $SV(i, j, l) = \max[S(i, j, l), G]$, and starting again from maturity, where the policy value is given by

$$V(n, j, l) = \max\left[S(n, j, l), G\right],$$

we compute the policy fair value in state of nature (i, j, l) by

$$V(i, j, l) = \max\left\{\frac{1}{4}e^{-r\Delta t}\left[V(i+1, j+1, l+1) + V(i+1, j+1, l) + V(i+1, j, l+1) + V(i+1, j, l)\right], SV(i, j, l)\right\}.$$

Numerical results and conclusions

European call option prices

$K = 1, r = 0.1, \sigma^2 = 0.4, n = 2000$

T	S_0	Model	$\delta = -0.5$	$\delta = -0.25$	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$
0.5	0.9	BL	0.125447	0.136115	0.139397	0.136252	0.126725
		Exp.	0.125427	0.136105	0.139397	0.136253	0.126729
		%	1.6×10^{-4}	7.4×10^{-5}	0	7.3×10^{-6}	3.2×10^{-5}
	1	BL	0.183572	0.194632	0.198085	0.194727	0.184445
		Exp.	0.183547	0.194620	0.198067	0.194728	0.184450
		%	1.4×10^{-4}	6.2×10^{-5}	9.1×10^{-5}	5.1×10^{-6}	2.7×10^{-5}
	1.1	BL	0.250935	0.261761	0.265182	0.261808	0.251371
		Exp.	0.250906	0.261746	0.265172	0.261811	0.251379
		%	1.2×10^{-4}	5.7×10^{-5}	3.8×10^{-5}	1.2×10^{-5}	3.2×10^{-5}
1	0.9	BL	0.202451	0.217075	0.221547	0.217298	0.204490
		Exp.	0.202421	0.217064	0.221543	0.217305	0.204499
		%	1.5×10^{-4}	4.1×10^{-5}	1.8×10^{-5}	3.2×10^{-5}	4.4×10^{-5}
	1	BL	0.267154	0.282166	0.286810	0.282350	0.268819
		Exp.	0.267119	0.282153	0.286792	0.282358	0.268831
		%	1.3×10^{-4}	4.6×10^{-5}	6.3×10^{-5}	2.8×10^{-5}	4.5×10^{-5}
	1.1	BL	0.338071	0.353032	0.357682	0.353171	0.339331
		Exp.	0.338030	0.353015	0.357688	0.353181	0.339345
		%	1.2×10^{-4}	4.8×10^{-5}	1.7×10^{-5}	2.8×10^{-5}	4.1×10^{-5}
5	0.9	BL	0.522367	0.544690	0.551284	0.545302	0.527951
		Exp.	0.522346	0.544784	0.551409	0.545452	0.528071
		%	4.0×10^{-5}	1.7×10^{-4}	2.3×10^{-4}	2.8×10^{-4}	2.3×10^{-4}
	1	BL	0.606416	0.629207	0.635922	0.629801	0.611836
		Exp.	0.606391	0.629309	0.636113	0.629967	0.611969
		%	4.1×10^{-5}	1.6×10^{-4}	3.0×10^{-4}	2.6×10^{-4}	2.2×10^{-4}
	1.1	BL	0.692289	0.715385	0.722274	0.715957	0.697517
		Exp.	0.692260	0.715495	0.722430	0.716139	0.697663
		%	4.2×10^{-5}	1.5×10^{-4}	2.2×10^{-4}	2.5×10^{-4}	2.1×10^{-4}

European and American put option prices

$K = 100, r = 0.03, n = 2000$

T	S_0	Model	$\delta = -0.699, \sigma = 0.238$	$\delta = -0.615, \sigma = 0.095$	$\delta = -0.985, \sigma = 0.222$
1	90	BL	11.296647	7.778738	8.765310
		Exp.	11.297032	7.779064	8.764776
		BL-A	13.227240	10.225982	12.414099
	100	BL	6.324898	1.992498	4.058787
		Exp.	6.325098	1.992518	4.058609
		BL-A	7.305797	2.512294	5.486579
	110	BL	3.285257	0.262912	1.759613
		Exp.	3.285260	0.262970	1.759745
		BL-A	3.726910	0.305824	2.197102
5	90	BL	13.296594	4.705028	8.409075
		Exp.	13.296671	4.705067	8.408492
		BL-A	17.941174	10.365024	15.249401
	100	BL	10.081635	2.052038	5.779318
		Exp.	10.081921	2.052072	5.779065
		BL-A	13.321288	3.852308	9.771197
	110	BL	7.638185	0.814903	3.998993
		Exp.	7.637898	0.814994	3.999004
		BL-A	9.897899	1.334027	6.309949
10	90	BL	12.824455	2.845119	7.220350
		Exp.	12.824871	2.845127	7.219889
		BL-A	20.247819	10.430872	16.630543
	100	BL	10.502314	1.445752	5.478781
		Exp.	10.502171	1.445823	5.478575
		BL-A	16.119620	4.246666	11.651890
	110	BL	8.644152	0.716457	4.205275
		Exp.	8.644055	0.716569	4.205273
		BL-A	12.940724	1.744690	8.308656

Equity-linked policy values

$G = 10, K = 10, r = 0.03, n = 2000$

T	S_0	Model	$\delta = -0.699, \sigma = 0.238$	$\delta = -0.615, \sigma = 0.095$	$\delta = -0.985, \sigma = 0.222$
1	9	BL	10.129835	9.777930	9.876764
		Exp.	10.129703	9.777906	9.876478
		BL-S	10.189318	9.999850	9.999850
	10	BL	10.632679	10.199312	10.406138
		Exp.	10.632510	10.199252	10.405861
		BL-S	10.655137	10.225025	10.416775
	11	BL	11.328734	11.026359	11.176247
		Exp.	11.328526	11.026297	11.175975
		BL-S	11.337323	11.028053	11.179996
5	9	BL	10.330102	9.470633	9.841558
		Exp.	10.329667	9.470507	9.840849
		BL-S	14.125519	9.999250	13.032900
	10	BL	11.008655	10.205349	10.578655
		Exp.	11.008192	10.205207	10.577906
		BL-S	15.727756	10.327935	11.179996
	11	BL	11.764359	11.081650	11.400695
		Exp.	11.763790	11.081499	11.399900
		BL-S	17.330020	11.111923	16.004105
10	9	BL	10.283135	9.284703	9.723109
		Exp.	10.282487	9.284513	9.721989
		BL-S	19.877509	9.998500	19.690647
	10	BL	11.050998	10.144787	10.549072
		Exp.	11.050217	10.144582	10.547857
		BL-S	22.124030	10.545844	21.923893
	11	BL	11.865258	11.071879	11.421841
		Exp.	11.864406	11.071657	11.420527
		BL-S	24.370582	11.594710	24.157138

Concluding remarks

- We propose a lattice-based discretization of a sBm characterized by a skew-normal distribution, which allows the valuation of American-style contingent claims in addition to European options.
- It represents one of the first attempts to price American-type derivatives under a sBm (the only other contributions are due to **Hussain et al. (2023)** and **Hu et al. (2024)**), and its application may be useful both in financial and actuarial markets.
- The proposed approach is very simple to implement and very efficient when evaluating contingent claims from a twofold point of view: accuracy of the obtained results and speed of execution.
- Numerical experiments assess the model accuracy.
- Future works will be focused on the analysis of the extension of the proposed model to approximate correlated sBms and to price more complex contingent claims.

THANK YOU FOR YOUR ATTENTION!!!

References

Black, F. and M. Scholes (1973). The pricing of options and corporate liabilities. Journal of Political Economy 8, 637–654.

Brennan, M. J. and E. S. Schwartz (1976). The pricing of equity-linked life insurance policies with an asset value guarantee. Journal of Financial Economics 3(3), 195–213.

Cont, R. and P. Tankov (2004). Financial Modelling with Jump Processes.

Corns, T. R. A. and S. E. Satchell (2007). Skew Brownian motion and pricing European options. The European Journal of Finance 13(6), 523–544.

Cox, J. C. and M. Rubinstein (1985). Option Markets.

Hu, Y., B. Lindquist, S. Rachev, and F. Fabozzi (2024). Option pricing using a skew random walk binary tree. Journal of Risk and Financial Management 17(4).

Hussain, S., H. Arif, M. Noorullah, and A. A. Pantelous (2023). Pricing American options under Azzalini Ito-McKean skew Brownian motions. Applied Mathematics and Computation 451.

Itô, K. and H. McKean (1965). Diffusion Processes and Their Sample Paths.

Zhu, S. P. and X. J. He (2018). A new closed-form formula for pricing European options under a skew Brownian motion. The European Journal of Finance 24(12), 1063–1074.