Pricing insurance products in a stochastic correlation framework: A lattice approach

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Building resilience to emerging risks
in financial and insurance markets

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- when the dynamics of these two state variables are stochastic, both the discount factor and the survival probability become random
- under the assumption that interest and mortality variables are independent, the fair value of the pure endowment is expressed as the product of the expectations of the two random terms under the risk-neutral probability measure

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- the advent of financial innovations, along with rapid technological changes, ongoing regulation overhauls, natural disasters and catastrophes, and political events around the globe, simultaneously affect both financial and mortality risks

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- the advent of financial innovations, along with rapid technological changes, ongoing regulation overhauls, natural disasters and catastrophes, and political events around the globe, simultaneously affect both financial and mortality risks
- thus, an integrated modelling that includes a description and quantification of the risks' dependence is of high importance

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- the first attempt to consider the dependence between financial and demographic factors, when evaluating life insurance policies, dates back to Jalen et al (2009) extended by Liu et al (2014) based on the change of numeraire technique
- the above mentioned papers model the dependence by considering stochastic demographic and financial factors with constant correlation only.

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- our motivation is to improve the actuarial framework, where the financial and demographic factors have a correlation whose movement is in accord with a bounded Jacobi process or with a transformed modified Ornstein-Uhlenbeck process

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- thus, the practicability of the discrete-time lattice-based approach as a modelling tool is even further heightened
- moreover, its significance is brought to the fore especially under the presence of product-embedded guarantees, like a surrender option having an American-style option element, in as much as the distribution of the optimal exercise time is not known.

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- then we merge together the three univariate lattices ad we obtain a trivariate lattice which approximate the joint evolution of the three processes
- the probability associated with each branch of the trivariate lattice is defined in order to replicate the varying correlation value affecting the interest rate and mortality intensity
- the numerical investigations delving into various aspects of model validation shows the merits of our new lattice development custom-built for dependent risk factors with stochastic correlation

The framework

- Under $\mathbb{Q}$, we suppose that the interest rate, r_t, and the mortality intensity for individual of age x, μ_t^x, are defined as,

$$dr_t = \kappa_r(\theta_r - r_t)dt + \sigma_r dW_t^1$$

$$d\mu_t^x = c_{\mu^x}\mu_t^x dt + \sigma_{\mu^x} W_t^2$$

the Brownian motions W_t^1 and W_t^2 are correlated with correlation coefficient having value ρ_t at time t that is modelled under $\mathbb{Q}$ through the bounded Jacobi process

$$d\rho_t = \kappa_\rho(\theta_\rho - \rho_t)dt + \sigma_\rho\sqrt{1-\rho_t^2}dW_t^3$$

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- the interest rate process is approximated through a recombining binomial lattice
- in general, at i-th time interval, the nodes values are determined as:
- on the highest path, $r(i,i) = r(i-1,i-1) + \sigma_r\sqrt{\Delta t}$

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- to complete the process approximation, we are left to define the jump probability for each node (i,j)
- the probability of an upward jump, $p_r(i,j)$, is defined in order to match the first two local order moments of the target continuous-time distribution, at least within the limit, i.e.,

$$p_r(i,j) = \frac{r(i,j) + \kappa_r(\theta_r - r_t)\Delta t - r(i+1,j)}{r(i+1,j+1) - r(i+1,j)},$$

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- a similar lattice procedure is also followed to discretize the correlation process. In this case we cut the lattice at -1 from bottom and at 1 from top
- upward jump probabilities, $p_\mu(i, l)$ and $p_\rho(i, h)$, and downward jump probabilities, $q_\mu(i, j) = 1 - p_\mu(i, l)$ and $q_\rho(i, h) = 1 - p_\rho(i, h)$ are defined as before to guarantee the matching of the first two local order moments of the target continuous-time distribution, at least within the limit

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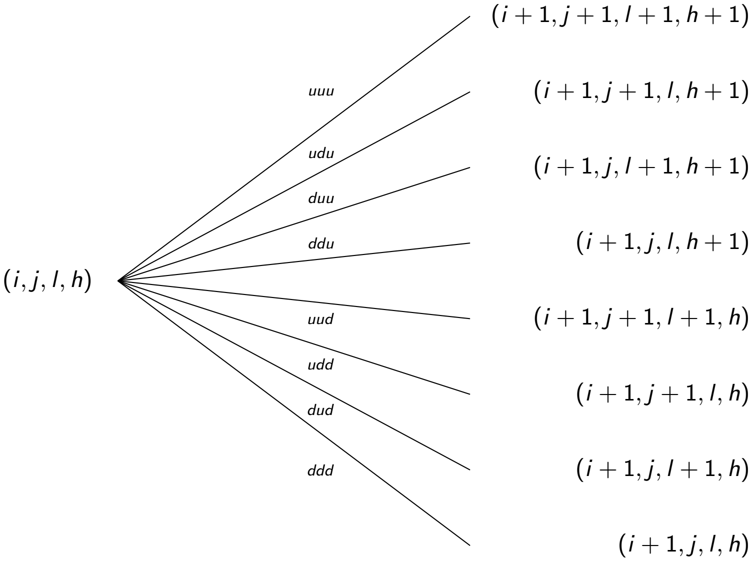
- from (i, j, l, h), eight branches arise, each of which represents a scenario of the possible process outcomes
- for example, $(i+1, j+1, l+1, h+1)$ represents the state of nature where the three processes jump up, $(i+1, j+1, l, h+1)$ the state of nature where both the short rate and the correlation jump up while the intensity of mortality jumps down, and so on

The framework



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- The transition probabilities associated with the scenarios generated starting from state of nature (i, j, l, h), labelled by p_{uuu}, p_{uud}, p_{udu}, p_{udd}, p_{duu}, p_{dud}, p_{ddu}, and p_{ddd}, are obtained by solving a linear system in which we impose that the matching of the marginal transition probabilities characterizing the movements in the lattice for r_t, μ_t^x, and ρ_t, and the correlation affecting the involved processes

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- the obtained solution is the following:

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\begin{aligned}
p_{uuu} &= p_r(i,j)p_\mu(i,l)p_\rho(i,h) + \frac{\rho(i,h)}{8};\\
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Applications

- We compute the fair value of a mortality bond, issued at time t, paying fixed coupons, C, at regular intervals $t+z, z=1,\dots,N$, and a random principal, $L(T)$, at maturity $t+N$ linked to longevity or mortality experiences

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- the mortality bond price at time t under $\mathbb{Q}$ is given by

$$B_M(t,T) = C\sum_{z=1}^{N} P(t,t+z) + \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du} L(T)\right],$$

where $P(t,t+z)$ is the value at time t of a zero coupon bond with maturity $t+z$, i.e., $P(t,t+z) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t+z} r(u)du}\right]$.

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- consider a cohort aged x at time t and define $L(T) = Kl(T)$, where K is the bond face value and $l(T)$ is the loss generated by the comparison between p_T, i.e., the realized survivance between age x and age $x + (T - t)$, and p_t, i.e., the fixed reference value

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- in particular, we consider a product that protects the issuer against possible deviation of the observed mortality with respect to a fixed reference value
- consider a cohort aged x at time t and define $L(T) = KI(T)$, where K is the bond face value and $I(T)$ is the loss generated by the comparison between p_T, i.e., the realized survivance between age x and age $x + (T - t)$, and p_t, i.e., the fixed reference value
- the loss $I(T)$ has the form $I(T) = 1 + \lambda(p_T - p_t)$, where $\lambda \in (0, 1]$ is a correction term governing the recovery of the survival spread

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- the value of the mortality bond at time t is computed as

$$B_M(t,T) = C\sum_{z=1}^{N} P(t,t+z) + (K - \lambda K p_t)\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}\right] +$$

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- The valuation of the last addendum in is done recursively on the trivariate lattice imposing at maturity $B(n,j,l,h) = \lambda K$

$$B(i,j,l,h) = e^{-[r(i,j)+\mu^x(i,l)]\Delta t}[p_{uuu}B(i+1,j+1,l+1,h+1) + p_{uud}B(i+1,j+1,l+1,h) + p_{udu}B(i+1,j+1,l,h+1) + p_{udd}B(i+1,j+1,l,h) + p_{duu}B(i+1,j,l+1,h+1) + p_{dud}B(i+1,j,l+1,h) + p_{ddu}B(i+1,j,l,h+1) + p_{ddd}B(i+1,j,l,h)]$$

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T	TL	MC (SE)	%
2	91.003766	91.04429 (0.034)	4.45E-04
5	72.359304	72.38516 (0.107)	3.57E-04
10	61.287339	61.12100 (0.405)	2.72E-03

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- in particular we fix the attention on the policy with time to maturity $T = 5$ years and compute the mortality bond value for three different initial correlation value ρ_0, i.e., -0.3, 0, and 0.3, when varying the value of the parameter θ_ρ in the stochastic correlation process that ranges in the interval $[-0.3, 0.3]$, by considering a step of 0.1

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- the computed absolute difference values that ranges from 0.01% to 2.60% evidence the impact on the mortality bond values impressed by the stochastic correlation

Applications

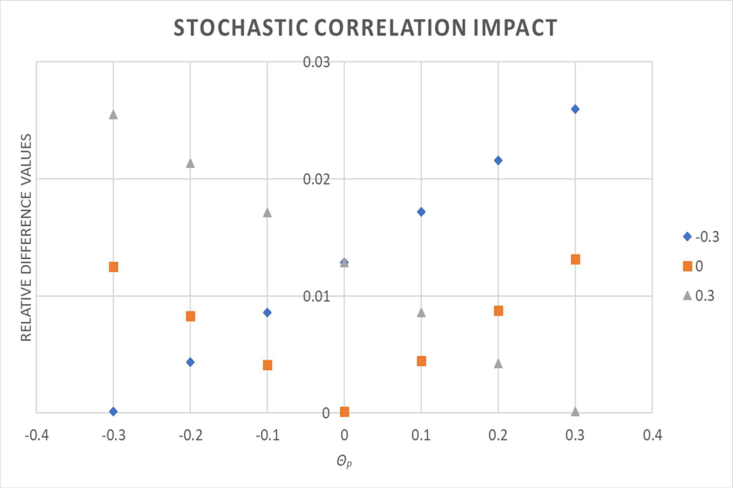

Figure: **Relative difference values.**

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The End

Thank You for Your attention!