On variable annuities with surrender charges

Gabriele Stabile
Sapienza University of Rome - MEMOTEF
joint paper with **T. De Angelis and A. Milazzo**
University of Turin - ESOMAS

Workshop Prin 2022

Building resilience to emerging risk in financial and insurance markets

Diamante (CS), June 12 and 13 2025

Finanziato dall'Unione europea
NextGenerationEU
Ministero dell'Università e della Ricerca
Italiadomani
PIANO NAZIONALE DI RIPRESA E RESILIENZA
SAPIENZA
UNIVERSITÀ DI ROMA

Table of Contents

Variable annuities (VA)

VA are insurance contracts designed to meet retirement and long-term investment goals.

Variable annuities (VA)

VA are insurance contracts designed to meet retirement and long-term investment goals.

The initial premium is invested in a selection of investment funds

VA offer life insurance and financial benefits:

- death benefit
- minimum interest rate guaranteed
- surrender option

Variable annuities (VA)

VA are insurance contracts designed to meet retirement and long-term investment goals.

The initial premium is invested in a selection of investment funds

VA offer life insurance and financial benefits:

- death benefit
- minimum interest rate guaranteed
- surrender option $\Longrightarrow$ surrender risk

These benefits are funded via a fee paid out of VA account

Variable annuities (VA)

VA are insurance contracts designed to meet retirement and long-term investment goals.

The initial premium is invested in a selection of investment funds

VA offer life insurance and financial benefits:

- death benefit
- minimum interest rate guaranteed
- surrender option $\Longrightarrow$ surrender risk

These benefits are funded via a fee paid out of VA account

Approaches in the literature

- intensity-based surrender (Baione et al. 2023, Ballotta et al. 2020, Russo et al. 2017)
- behavioral (Costabile et al. 2020, De Giovanni 2010, Li and Szimayer 2014)
- from a financial perspective (Bacinello et al. 2001-2024, Bernard et al. 2014, Mackay 2017, Jeon and Kwak 2021)

Variable annuities (VA)

VA are insurance contracts designed to meet retirement and long-term investment goals.

The initial premium is invested in a selection of investment funds

VA offer life insurance and financial benefits:

- death benefit
- minimum interest rate guaranteed
- surrender option $\implies$ surrender risk

These benefits are funded via a fee paid out of VA account

Approaches in the literature

- intensity-based surrender (Baione et al. 2023, Ballotta et al. 2020, Russo et al. 2017)
- behavioral (Costabile et al. 2020, De Giovanni 2010, Li and Szimayer 2014)
- from a financial perspective (Bacinello et al. 2001-2024, Bernard et al. 2014, Mackay 2017, Jeon and Kwak 2021)

Our research would provide a theoretical analysis of VA with a focus on surrender risk.

The model

- VA account

$$\begin{cases} \mathrm{d}X_t = X_t\big[(r-c)\mathrm{d}t + \sigma \mathrm{d}W_t\big], \\ X_0 = x_0 > 0, \end{cases}$$

where $c \geq 0$ denotes the constant fee rate and $x_0 > 0$ is the initial premium

The model

- VA account

$$\begin{cases} \mathrm{d}X_t = X_t\big[(r-c)\mathrm{d}t + \sigma \mathrm{d}W_t\big], \\ X_0 = x_0 > 0, \end{cases}$$

  where $c \geq 0$ denotes the constant fee rate and $x_0 > 0$ is the initial premium
- The probability that a person alive at time t survives the next z years is:

$$_z p_{\eta+t} := \mathbb{P}(\tau_d > t + z | \tau_d > t) = \exp\left(-\int_t^{t+z} \mu(\eta+s)ds\right)$$

  where $\mu : [0,+\infty) \to [0,+\infty)$ is the mortality force (deterministic function)

VA payoff

Consider a VA contract with maturity T and let τ_d be the residual lifetime of an individual of age η

- Maturity benefit

$$\max\left\{x_0 e^{gT}, X_T\right\},$$

  with $g \geq 0$ denoting the guaranteed minimum interest rate per annum

VA payoff

Consider a VA contract with maturity T and let τ_d be the residual lifetime of an individual of age η

- Maturity benefit

$$\max\left\{x_0 e^{gT}, X_T\right\},$$

with $g \geq 0$ denoting the guaranteed minimum interest rate per annum

- Death benefits: in case $\tau_d < T$

$$\max\left\{x_0 e^{g\tau_d}, X_{\tau_d}\right\}$$

VA payoff

Consider a VA contract with maturity T and let τ_d be the residual lifetime of an individual of age η

- Maturity benefit
$$\max\left\{x_0 e^{gT}, X_T\right\},$$
with $g \geq 0$ denoting the guaranteed minimum interest rate per annum
- Death benefits: in case $\tau_d < T$
$$\max\left\{x_0 e^{g\tau_d}, X_{\tau_d}\right\}$$
- Early surrender: if the policyholder decides to surrender the contract at any time $\tau \leq T$, she receives
$$\big(1 - k(\tau)\big)X_\tau,$$
where the penalty function $t \mapsto k(t)$ is non-increasing and twice continuously differentiable in $[0, T]$ with $k(T) = 0$

VA value

- At time 0 the value of the VA contract is

$$V_0 = \sup_{0 \le \tau \le T} \mathrm{E}\Big[\mathbb{1}_{\{\tau < \tau_d \wedge T\}} e^{-r\tau} \big(1 - k(\tau)\big) X_\tau + \mathbb{1}_{\{\tau \ge \tau_d \wedge T\}} e^{-r(\tau_d \wedge T)} \max\{x_0 e^{g(\tau_d \wedge T)}, X_{\tau_d \wedge T}\} \Big]$$

VA value

- At time 0 the value of the VA contract is

$$V_0 = \sup_{0\leq\tau\leq T} \mathrm{E}\Big[\mathbb{1}_{\{\tau<\tau_d\wedge T\}} e^{-r\tau}\big(1-k(\tau)\big)X_\tau + \mathbb{1}_{\{\tau\geq\tau_d\wedge T\}} e^{-r\,(\tau_d\wedge T)} \max\{x_0 e^{g(\tau_d\wedge T)}, X_{\tau_d\wedge T}\}\Big]$$

- Using independence between financial and demographic uncertainties

$$V_0 = \sup_{0\leq\tau\leq T} \mathrm{E}\Big[\int_0^\tau e^{-rt}\,{}_tp_0\,\mu(t)\max\{x_0 e^{gt}, X_t\}\mathrm{d}t + \mathbb{1}_{\{\tau<T\}} e^{-r\tau}\,{}_\tau p_0\big(1-k(\tau)\big)X_\tau + \mathbb{1}_{\{\tau=T\}} e^{-r\,T}\,{}_T p_0 \max\{x_0 e^{gT}, X_T\}\Big]$$

Continuation and stopping regions

- Continuation region

$$\mathcal{C} = \{(t, x) \in [0, T) \times (0, \infty) : x < \ell_{x_0}(t)\},$$

- Surrender region

$$\mathcal{D} = \{(t, x) \in [0, T) \times (0, \infty) : x \geq \ell_{x_0}(t)\} \cup \big(\{T\} \times (0, \infty)\big).$$

- Optimal surrender time reads

$$\tau_* = \inf\{s \in [0, T) : X_s^{x_0} \geq \ell_{x_0}(s)\} \wedge T.$$

Pricing formula

Set $f(t) := k(t)(c + \mu(t)) - \dot{k}(t) - c$ and define

$$t^* := \inf\{t \in [0, T) : f(t) < 0\} \wedge T, \quad \text{with } \inf \varnothing = \infty$$

For each $x_0 \in (0, \infty)$ there is a function $\ell_{x_0} : [0, T] \to [0, \infty]$, where $\ell_{x_0}(t) = +\infty$ for $t \in [0, t^*)$ and $\ell_{x_0} \in C([t^*, T])$ with $\ell_{x_0}(T) = x_0 e^{gT}$, such that the VA's rational price (at time zero) reads

$$V_0 = e^{-rT}\,{}_T p_0 \mathrm{E}\left[\max\{x_0 e^{gT}, X_T^{x_0}\}\right] + \int_0^T e^{-rs}\,{}_s p_0 (\mu(s) - f(s)) \mathrm{E}\left[X_s^{x_0} \mathbb{1}_{\{\ell_{x_0}(s) \le X_s^{x_0}\}}\right] \mathrm{d}s$$
$$+ \int_0^T e^{-rs}\,{}_s p_0 \mu(s) \mathrm{E}\left[\max\{x_0 e^{gs}, X_s^{x_0}\} \mathbb{1}_{\{\ell_{x_0}(s) > X_s^{x_0}\}}\right] \mathrm{d}s$$

Optimal surrender boundary

The optimal surrender boundary is obtained as $\ell_{x_0}(t) := x_0 e^{gt}/b(t)$, where b is the unique solution of the integral equation

$$e^{-c(T-t)}\,{}_{T-t}p_t \int_1^{\infty} \Phi\big(b(t), T-t, y\big)\,(y-1)\,\mathrm{d}y$$
$$+\int_t^T e^{-c(s-t)}\,{}_{s-t}p_t \int_{b(s)}^{\infty} \big(\mu(s)(y-1)^+ + f(s)\big)\Phi\big(b(t), s-t, y\big)\,\mathrm{d}y\,\mathrm{d}s = 0,$$

where

$$\Phi(z,s,y) = \frac{1}{\sqrt{2\pi\sigma^2 s}\,y}\exp\left(-\frac{\left(\ln\frac{y}{z} - \left(\alpha - \frac{\sigma^2}{2}\right)s\right)^2}{2\sigma^2 s}\right)$$

Model parameters

- Gompertz-Makeham mortality model

$$\mu(t) = A + BC^t,$$

with $A = 0.0001$, $B = 0.00035$ and $C = 1.075$ (MacKay et al 2017)

Model parameters

- Gompertz-Makeham mortality model

$$\mu(t) = A + BC^t,$$

with $A = 0.0001$, $B = 0.00035$ and $C = 1.075$ (MacKay et al 2017)

- Exponential penalty function (Hilpert 2020, Jeon and Kwak 2021)

$$k(t) = 1 - e^{-K(T-t)},$$

where $K > 0$ is the surrender charge intensity

Model parameters

- Gompertz-Makeham mortality model

$$\mu(t) = A + BC^t,$$

with $A = 0.0001$, $B = 0.00035$ and $C = 1.075$ (MacKay et al 2017)

- Exponential penalty function (Hilpert 2020, Jeon and Kwak 2021)

$$k(t) = 1 - e^{-K(T-t)},$$

where $K > 0$ is the surrender charge intensity

- Benchmark parameters

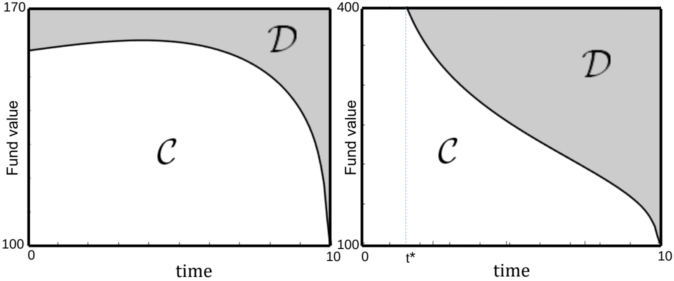
$$\begin{cases} T = 10 \text{ years}, \ \eta = 50 \text{ years, initial fund value } x_0 = 100, \ K = 1.4\%, \\ \text{annual risk-free rate } r = 5\%, \text{ annual volatility } \sigma = 20\%, \\ \text{annual fee rate } c = 2.5\%, \text{ minimum guaranteed } g = 0\%. \end{cases}$$

Sensitivity to the surrender charge intensity

The contract is never surrendered if $K \geq c$. In particular, $K = c$ is the minimum surrender charge intensity that totally removes the incentive to an early surrender

Sensitivity to the surrender charge intensity

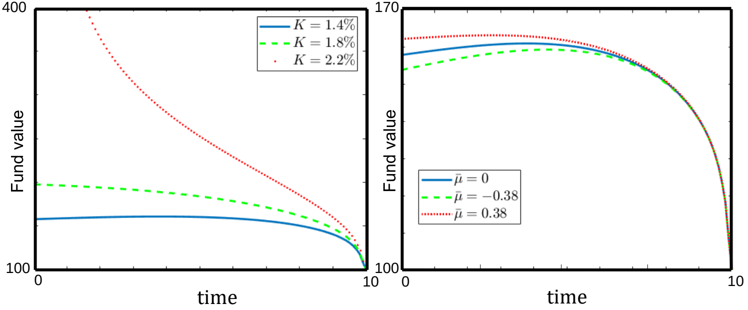
The contract is never surrendered if $K \geq c$. In particular, $K = c$ is the minimum surrender charge intensity that totally removes the incentive to an early surrender



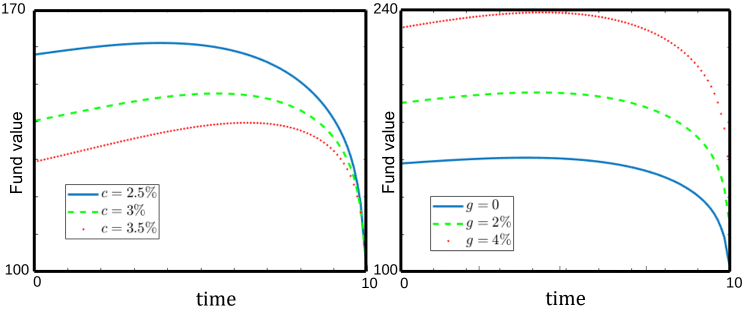
$K = 1.4\%$ (left plot) whereas $K = 2.2\%$ (right plot)

Comparing effects of surrender charge and mortality

Proportional hazard rate transformation: $(1+\bar{\mu})\mu(\cdot)$. For $\bar{\mu} \in \{0, -0.38, 0.38\}$ the life expectancy of a 50 years old individual is respectively 21.7, 26.6 and 18.5 years



Sensitivity to fee and minimum guaranteed rates



Fair fee

Consider the contract's value $V_0 = v(0, x_0)$ as a function of the fee rate $c \mapsto V_0(c) = v(0, x_0; c)$, we are after the fee rate c^* for which $V_0(c^*) = x_0$

Age at issue	50	60	70
Fair fee	2%	2.2%	2.5%

Sensitivity to risk-free and minimum guaranteed rates

- Price at time zero of a VA *without* early surrender option

$$U_0 := \mathrm{E}\Big[e^{-r\,(\tau_d\wedge T)}\max\{x_0 e^{g(\tau_d\wedge T)}, X_{\tau_d\wedge T}\}\Big],$$

- Scenarios:
  scenario A: $c = 4\%$, $K = 1.8\%$
  scenario B: $c = 4\%$, $K = 1.4\%$
  scenario C: $c = 2.5\%$, $K = 1.8\%$
  scenario D: $c = 2.5\%$, $K = 1.4\%$

Sensitivity to risk-free and minimum guaranteed rates

Spread	Scenario	V_0	U_0	V^{SO}
$r-g=5\%$	A	87.2	82.7	4.5
	B	89.07	82.7	6.37
	C	90.97	89.96	1.01
	D	92.16	89.96	2.2
$r-g=3\%$	A	93.37	90.56	2.81
	B	94.52	90.56	3.96
	C	97.44	96.75	0.69
	D	98.2	96.75	1.45
$r-g=1\%$	A	103.38	101.7	1.68
	B	104.04	101.7	2.34
	C	107.17	106.71	0.46
	D	107.64	106.71	0.93

Thank you for your attention!

A numerical scheme for the boundary

Take an equally-spaced partition $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T$ with $\Delta t := t_{i+1} - t_i = T/n$

The optimal surrender boundary is given by $\ell_{x_0}(t) = x_0 e^{gt}/b(t)$

Let $b^k(t_j)$, for $j = 0, 1, \ldots n$, be the values of the boundary obtained after the k-th iteration. Then, the value $b^{k+1}(t_j)$ for the $(k+1)$-th iteration is computed as the value θ that solves the following discretised equation:

$$e^{-c(T-t_j)}\,{}_{T-t_j}p_{t_j} \int_1^{\infty} \Phi(\theta, T - t_j, y)\mathrm{d}y \\ + \Delta t \sum_{i=j+1}^{n} e^{-c(t_i - t_j)}\,{}_{t_i - t_j}p_{t_j} \int_{b^k(t_i)}^{\infty} \Phi(\theta, t_i - t_j, y) H(t_i, y)\mathrm{d}y = 0,$$

where $\Phi(z, s, y)$ is the log-normal density and $H(t, x) = \mu(t)(x-1)^+ + f(t)$.
Given a tolerance parameter $\varepsilon > 0$, the algorithm stops when

$$\max_{j=0,1,\ldots,n} |b^k(t_j) - b^{k+1}(t_j)| < \varepsilon.$$