Minimum capital requirement for non-life insurance with risk budgeting portfolios

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Outline

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Rationale behind the model

- Insurance regulations, such as Solvency II in the European Union and the Swiss Solvency Test, require insurance companies to set aside capital to protect themselves against unexpected losses over a defined solvency horizon.
- Risk over this horizon is typically measured using Value-at-Risk (VaR) or Conditional Value-at-Risk (CVaR).
- When computing capital requirements, investments in financial markets must be considered to accurately assess the insurer's financial position at the end of the solvency horizon.
- The risk budgeting approach enables the selection of portfolios in which the risk contribution of each asset is predetermined. It has low sensitivity to input parameter perturbations and often outperforms traditional strategies.
- The proposed model integrates risk budgeting with capital requirement computation for non-life insurers.

Outline

Insurer's net loss over the solvency horizon τ

- Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and n financial assets. Let Z denote the insurer's liability over τ, and let $\mathbf{R} = (R_1, \ldots, R_n)^\top$ be the vector whose ith component, R_i, represents the gross return over τ for the ith asset.
- The insurer's net loss over the solvency horizon is given by

$$L(c, \mathbf{x}) = Z - (p + c)\mathbf{R}^\top \mathbf{x},$$

  where:
  - p is the premium collected from policyholders at the inception of the horizon, which is available for investment;
  - c is the solvency capital that must be determined at the beginning of τ to satisfy a regulatory constraint;
  - $\mathbf{x} = (x_1, \ldots, x_n)^\top$ represents the portfolio weights, with x_i denoting the proportion invested in the ith asset.

Computation of $\mathbf{x}$ and c

- We consider the following optimization problem:

$$
\begin{aligned}
\min_{\mathbf{x},c} \quad & c \\
\text{s.t.} \quad & \mathrm{CVaR}_\alpha(L(c,\mathbf{x})) \le 0, \\
& \mathbf{x} \ge \mathbf{0},\ \mathbf{1}^\top \mathbf{x} = 1, \\
& c \ge 0.
\end{aligned}
$$

- Instead of computing the optimal pair $(\mathbf{x}^*, c^*)$ simultaneously, as in Asanga et al. 2014 and Staino et al. 2023, we decompose the problem of determining the portfolio weights $\mathbf{x}^*$ and the capital c^* into two stages:
  - First, we determine the portfolio weights $\mathbf{x}^*$ by applying risk budgeting conditions to the CVaR of the negative portfolio log-return.
  - Then, given $\mathbf{x}^*$, we compute the capital requirement c^* such that the CVaR of L does not exceed zero.

Risk budgeting approach

- Given the relative risk budgets $b_1, \ldots, b_n$ such that $b_i > 0$, for $i = 1, \ldots, n$, and $\sum_{i=1}^n b_i = 1$, the risk budgeting portfolio $\mathbf{x}^*$ is given by

$$x_i^* = \frac{y_i^*}{\sum_{i=1}^n y_i^*}, \quad i = 1, \ldots, n.$$

  where $\mathbf{y}^*$ is the solution of

$$\begin{aligned} \min_{\mathbf{y}} \quad & \mathrm{CVaR}_\alpha \left(-\sum_{i=1}^n y_i \log(R_i) \right) \\ \text{s.t.} \quad & \sum_{i=1}^n b_i \ln y_i \geq 0. \end{aligned}$$

- The portfolio $\mathbf{x}^*$ satisfies the risk budgeting constraints:

$$b_i \, \mathrm{CVaR}_\alpha \left(-\sum_{i=1}^n x_i^* \log(R_i) \right) = x_i^* \frac{\partial}{\partial x_i} \mathrm{CVaR}_\alpha \left(-\sum_{i=1}^n x_i^* \log(R_i) \right),$$

  for $i = 1, \ldots n$.

Risk budgeting approach (cont'd)

- For an integrable random loss L, we have

$$\mathrm{CVaR}_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_t(L)dt = \inf_{s\in\mathbb{R}}\left\{s + \frac{1}{1-\alpha}E\left[(L-s)_+\right]\right\}.$$

- To compute $\mathbf{y}^*$, we generate m scenarios $\mathbf{r}_j = (r_{1,j},\ldots,r_{n,j})'$ of the asset gross returns and solves the optimization problem

$$\min_{s,\mathbf{y}} \quad s + \frac{1}{m(1-\alpha)}\sum_{j=1}^{m}\left(\ell^{(j)}(\mathbf{y}) - s\right)_+$$

$$\text{s.t.} \quad \sum_{i=1}^{n} b_i \ln y_i \geq 0,$$

  where $\ell^{(j)}(\mathbf{y}) = -\sum_{i=1}^{n} y_i \log(r_{i,j})$ is the jth loss scenario.

- Given $\mathbf{y}^*$, we rescale it to obtain the optimal portfolio weights $\mathbf{x}^*$ for the relative risk budgets $b_1,\ldots,b_n$.

CVaR regulatory constraint

- The CVaR regulatory constraint requires that the CVaR at level α of the insurer's net loss over the solvency horizon be less than or equal to zero:

$$\begin{aligned} \min_{c} \quad & c \\ \text{s.t.} \quad & \mathrm{CVaR}_\alpha(L(c, \mathbf{x}^*)) \leq 0, \\ & c \geq 0. \end{aligned}$$

- To solve this problem, we use the same m scenarios $\mathbf{r}_j = (r_{1,j}, \ldots, r_{n,j})^\top$ of the asset gross returns previously employed to compute the risk budgeting portfolio $\mathbf{x}^*$ and resort to the semiparametric formulation proposed by Asanga et al. 2014:

$$\begin{aligned} \min_{s,c} \quad & c \\ \text{s.t.} \quad & s + \frac{1}{m(1-\alpha)} \sum_{j=1}^{m} E\left[\left(Z - (p+c)\mathbf{R}^\top \mathbf{x}^* - s\right)_+ \middle| \, \mathbf{R} = \mathbf{r}_j \right] \leq 0, \\ & c \geq 0. \end{aligned}$$

CVaR regulatory constraint (cont'd)

- Assuming Z to be independent of $\mathbf{R}$ yields

$$\begin{aligned} \min_{s,c} \quad & c \\ \text{s.t.} \quad & g(s,c) \leq 0, \\ & c \geq 0. \end{aligned}$$

where

$$g(s,c) = s + \frac{1}{m(1-\alpha)} \sum_{j=1}^{m} E\left[\left(Z - (p+c)\mathbf{r}_j^\top \mathbf{x}^* - s\right)_+\right],$$

which is a convex function.

- This optimization problem can be solved with the Kelley-Cheney-Goldstein (KCG) cutting-plane algorithm.

Liability modelling

We have $g(s,c) = s + \frac{1}{m(1-\alpha)} \sum_{j=1}^{m} h\left((p+c)\mathbf{r}_j^\top \mathbf{x}^* + s\right)$, where

- if Z is lognormally distributed:

$$h(l) = \begin{cases} \exp\left(\mu + \frac{\sigma^2}{2}\right) - l, & l \leq 0, \\ \exp\left(\mu + \frac{\sigma^2}{2}\right) \Phi\left(\frac{\mu - \ln(l) + \sigma^2}{\sigma}\right) - l\Phi\left(\frac{\mu - \ln(l)}{\sigma}\right), & l > 0. \end{cases}$$

- if Z is gamma distributed:

$$h(l) = \begin{cases} k\theta - l, & l \leq 0, \\ k\theta\left[1 - F_G(l; k+1, \theta)\right] - l\left[1 - F_G(l; k, \theta)\right], & l > 0. \end{cases}$$

- if Z is distributed as a mixture of Erlang distributions with a common scale parameter:

$$h(l) = \begin{cases} \theta \sum_{i=1}^{m} \alpha_i k_i - l, & l \leq 0, \\ \sum_{i=1}^{m} \alpha_i \left\{k_i\theta\left[1 - F_G(l; k_i+1, \theta)\right] - l\left[1 - F_G(l; k_i, \theta)\right]\right\}, & l > 0. \end{cases}$$

Outline

Financial data

- We consider portfolios composed of the S&P 500 index, the iShares Barclays 1-3 Year Treasury Bond ETF (SHY), and the iShares iBoxx $ Investment Grade Corporate Bond ETF (LQD).
- Daily prices from January 2010 to December 2020, for a total of 2768 observations, are used to compute daily log-returns.
- The resulting daily log-returns are divided into two samples:
  - Sample A includes data from January 2010 to December 2015 (the first six years);
  - Sample B includes data from January 2016 to December 2020 (the last five years) and is used for out-of-sample analysis.

Asset	Min.	Max.	Mean	S.D.	Skewness	Kurtosis
S&P 500	-0.12765	0.08968	0.00043	0.01106	-0.86342	19.33641
SHY	-0.00439	0.00544	0.00005	0.00059	0.53278	9.60533
LQD	-0.05132	0.07131	0.00024	0.00448	0.32077	58.12501

Table: *Descriptive statistics about daily log-returns from January 2010 to December 2020 for the assets S&P 500, SHY, and LQD.*

Insurer's liability data

- We use the dataset *danishuni* from the R package *CASdatasets* Dutang and Charpentier 2020, which contains 2167 fire losses (in millions of Danish krone) from January 1980 to December 1990, adjusted for inflation to reflect 1985 values.
- We convert these losses to millions of U.S. dollars, aggregate them monthly, and apply the annual inflation index so that: the first monthly loss reflects January 2010 values, the second reflects February 2010 values, and so on, up to the last monthly loss, which reflects December 2020 values.
- The resulting monthly losses are split into two samples: Sample A' contains data from January 2010 to December 2015; Sample B' contains data from January 2016 to December 2020.

Min.	Max.	Mean	S.D.	Skewness	Kurtosis
3.59614	69.15245	13.88274	9.52692	3.50261	19.23287

Table: *Descriptive statistics about the data set danishuni of the R package CASdatasets of Dutang and Charpentier (2020) after some adjustments to have monthly losses in millions of U.S. dollars and concerning the period January 2010 - December 2020.*

Parameter estimation for the liability distributions

- We use the maximum likelihood estimator (MLE) for the three theoretical distributions.
- For the mixture of Erlang distributions with a common scale parameter, we apply the approach proposed by Lee and Lin 2010, who developed a modified expectation-maximization (EM) algorithm.

	$\hat{\mu}$	$\hat{\sigma}$				Log L	KS test
Lognormal	2.3548	0.5253				225.3566	0.0612
	(0.0619)	(0.0438)					(0.9350)
	$\hat{k}$	$\hat{\theta}$				Log L	KS test
Gamma	3.3735	3.6486				231.4724	0.1033
	(0.5375)	(0.6269)					(0.3993)
	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{k}_1$	$\hat{k}_2$	$\hat{\theta}$	Log L	KS test
Mixture	0.9861	0.0139	5	33	2.2840	221.7991	0.0700
	(0.2028)	(0.0020)			(0.1561)		(0.8478)

Table: *Parameter estimates of the three distributions considered, based on Sample A'. The losses in Sample A' are adjusted to reflect year-2015 values.*

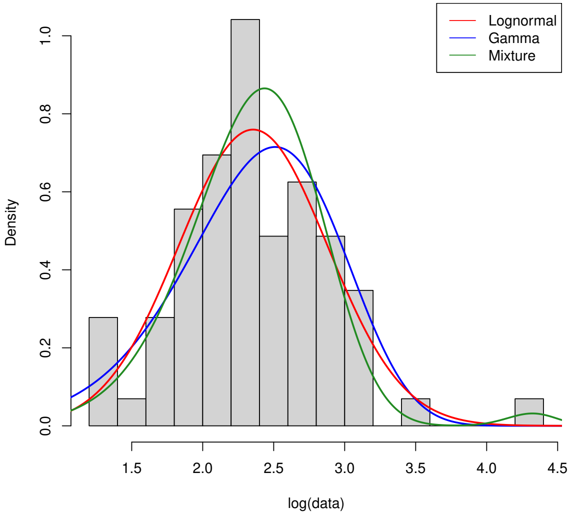

Figure: *Histogram of the log-transformed data for Sample A' with the addition of the theoretical curves. The losses in Sample A' are adjusted to reflect year-2015 values.*

Strategy to compute $\mathbf{x}^*$ and c^*

- We use $\alpha = 99\%$ as the confidence level for CVaR.
- We set $\tau = 21$ days as the solvency horizon, since the observed losses are on a monthly basis.
- At any date t, where an optimal decision $(\mathbf{x}^*, c^*)$ must be made, we apply the following strategy:
  1. Estimate the liability parameters for the three theoretical distributions unsing MLE.
  2. Apply the expected premium principle to compute the insurance premium: $p = (1+\eta)\,\mathbb{E}[Y]$, where the relative security loading factor η is 0.1.
  3. Generate $m = 10000$ scenarios $\mathbf{r}_j$, $j = 1, \ldots, m$, for asset gross returns using the moment-matching method of Høyland, Kaut, and Wallace 2003.
  4. Compute the portfolio weights $\mathbf{x}^*$ that satisfy risk budgeting parity, i.e., $b_i = 1/3$ for $i = 1, 2, 3$.
  5. Compute the optimal required capital c^* that ensures satisfaction of the CVaR regulatory requirement.

Values of $\mathbf{x}^*$ and c^* for Samples A and A'

		S&P 500	SHY	LQD
	c^*	x_1^*	x_2^*	x_3^*
Lognormal	29.987 (29.825)	0.0534 (0.4341)	0.8441 (0)	0.1025 (0.5659)
Gamma	23.970 (23.842)	0.0534 (0.3682)	0.8441 (0)	0.1025 (0.6318)
Mixture	67.614 (67.229)	0.0534 (0.7383)	0.8441 (0)	0.1025 (0.2617)

Table: *Values of the optimal capital requirement c^* and the risk parity weights $\mathbf{x}^*$. The values in brackets are those obtained with the model proposed by Staino et al. 2023. Computations are based on Sample A for asset log-returns and Sample A' for the insurer's liability. The losses in the sample A' are adjusted to reflect 2015 values.*

Out-of-sample analysis

- Using Samples A and A', we first compute the optimal solution $(c_0^*, \mathbf{x}_0^*)$ for the period $[0, \tau]$ by applying Steps 1-5 detailed above.
- We construct a new sample for asset log-returns by removing the first month's observations from Sample A and including those of the first month from Sample B, i.e., we carry out monthly portfolio rebalancing.
- For the insurer's liabilities, we build the new sample by retaining all observations from Sample A' and adding the first observation from Sample B'. We then adjust the losses in this updated sample to reflect values for the year of the most recent loss, which is 2016 in this case.
- Given the new samples for asset log-returns and losses, we recompute the optimal solution $(c_\tau^*, \mathbf{x}_\tau^*)$ for the next period $[\tau, 2\tau]$ by reapplying Steps 1–5.
- By repeating the sampling and optimization procedures until the end of Samples B and B', we obtain a sequence of optimal solutions $(c_{k\tau}^*, \mathbf{x}_{k\tau}^*)$, for $k = 0, 1, \ldots, K-1$, where K is the length of Sample B'.

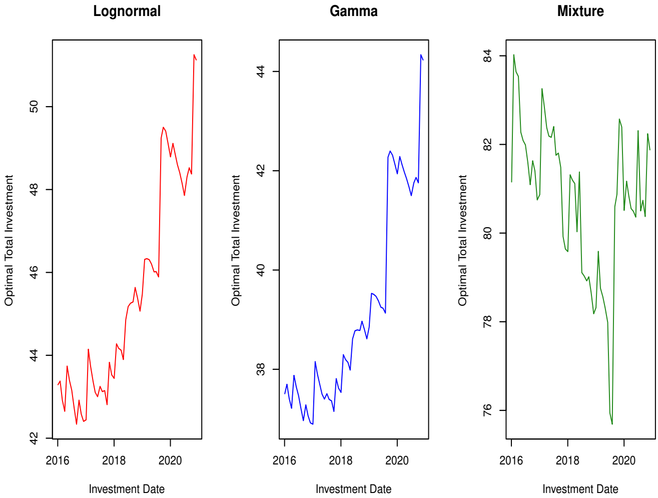

Figure: *Optimal total investments* $p_{k\tau} + c^*_{k\tau}$, $k = 0, 1, \ldots, K-1$.

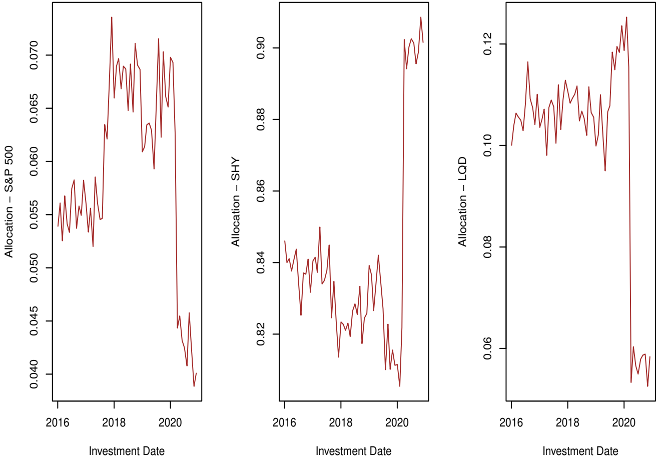

Figure: *Optimal asset allocations* $\mathbf{x}_k^*$, $k = 0, 1, \ldots, K-1$, *with risk parity*.

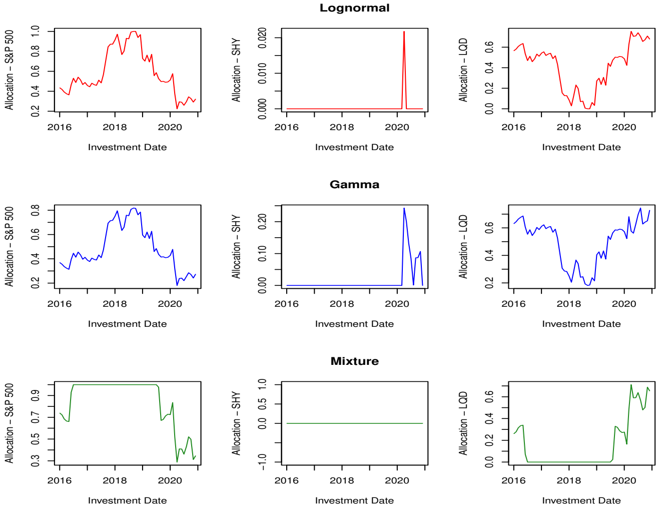

Figure: *Optimal asset allocations* $\mathbf{x}_k^*$, $k = 0, 1, \ldots, K-1$, *without risk parity.*

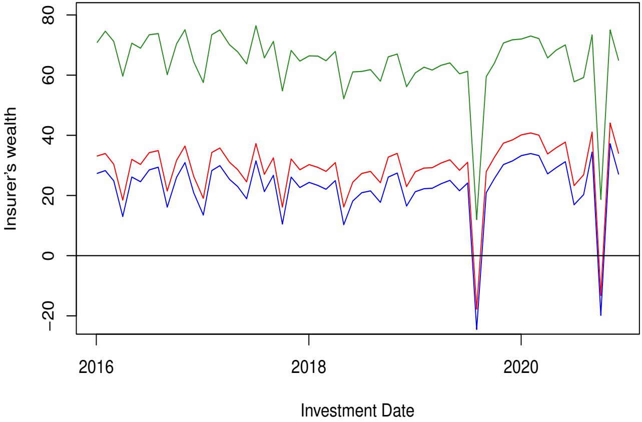

Figure: *Realized insurer's wealth,* $(p_{k\tau} + c^*_{k\tau})\mathbf{r}^\top_{k+1}\mathbf{x}^*_{k\tau} - z_{k+1}$, $k = 0, 1, \ldots, K-1$, *at the end of each month.*

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Conclusions

- The proposed model integrates a risk budgeting approach for asset portfolio selection with the computation of the solvency capital requirement for non-life insurance.
- The CVaR risk measure is employed both for the computation of risk parity portfolios and for determining the minimum capital that fulfils the solvency requirement.
- The analysis shows how the solvency capital requirement can be calculated under several loss distributions.
- The analysis does not aim to prove that one loss distribution is better than another. It simply shows that, for the considered dataset and with an expanding-window sampling of the insurer's losses, the Erlang mixture distribution can provide better protection compared to the lognormal and gamma distributions. However, this greater protection comes at a higher cost.

Thank you for your attention!

Questions or comments are welcome.

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