Guaranteed minimum income benefit valuation via a numéraire transformation approach

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Rogemar Mamon
Complex Adaptive Systems Lab
and
Centre for Multi-hazard Risk and Resilience
The University *of* Western Ontario, Canada

Joint work with
Yiming Huang (Department of Statistical and Actuarial Sciences, *The* University *of* Western Ontario, Canada)
Heng Xiong (Economics and Business School, Wuhan University, China)

Overview of this talk

Long-term investments or retirement-designed innovations

- Global appeal of variable annuities (VAs) $\rightarrow$ potential for enhanced investment outcomes through equity participation.
- Notable minimum guarantee riders:
  - death benefits (GMDB),
  - maturity benefits (GMMB),
  - accumulation benefits (GMAB),
  - **income benefits (GMIB)**,
  - and withdrawal benefits (GMWB)

  see Hardy (2003) and Ledlie et al. (2008) for a comprehensive discussion.
- Total annuity sales: \$385 billion in 2023 (LOMA Secure Retirement Institute).

Some literature on pricing & hedging guaranteed-maturity riders

Accurate valuation, **understanding risks**, and **hedging** - prime importance to insurers and regulators.

- **GMDB:** risk-neutral valuation (Milevsky & Posner, 2001); discounted density approach (Gerber et al., 2012); PDE-based method (Belanger et al., 2009); numerical-integration-based approach with surrender options (Shen et al., 2016)
- **GMMB:** regime-switching and stochastic mortality set up (Ignatieva et al., 2016); VIX-linked fee structure under a Heston volatility model (Cui et al., 2017).
- **GMAB:** three correlated risk factors (Huang et al., 2022)
- **GMWB:** pricing/hedging - financial economic perspective (Hyndman and Wenger [15]); valuation with step-up, bonus and surrender features in a low interest rate environment (Fontana and Rotondi, 2023).

Aims: **GMIB** pricing and risk analysis

GMIB is an attractive investment feature to policyholders. Reasons are:

(i) *Protection against longevity risk*
GMIB transfers longevity risk to insurers - option to convert retirement savings into a life annuity.

(ii) *Provision of stable payments irrespective of market performance*
GMIB ensures a guaranteed minimum income upon annuitisation, shielding policyholders from adverse impact of market conditions with a steady income stream during retirement.

(iii) *Equitable market participation, with downside protection*
Policyholders can capitalise on equity market growth and benefit from the security of a guaranteed minimum level of annuity payments.

(iv) *Transparency*
Predetermined guaranteed minimum payments at each age, making retirement planning endeavours simple.

Comparable product existing in European market

GMIB versus Guaranteed Annuity Option (GAO)

- There are similarities between GAO and GMIB.
- GAO's pricing and hedging have been extensively explored (e.g., Boyle & Hardy (2003); Liu et al. (2013 & 2014); Ballotta and Haberman (2003); Pelsser (2003); and Zhao et al. (2018), amongst others.)
- Both GAO and GMIB offer a guaranteed conversion rate upon annuitisation.
- *GMIB distinguishes itself from the GAO in terms of product design and benefit structure.*
- Principles under GAO's analysis cannot be readily extrapolated to the GMIB.
- For this reason, a distinct and targeted study is warranted specifically for the GMIB.

Some current literature on GMIB pricing

- Bauer et al. (2008) - a general framework for a variety of VA guarantees, with models for: investment (GBM) and mortality (deterministic).

- Marshall et al. (2010) - Hull-White interest rate dynamics, with GMIBs contract designs priced in a complete market covering financial risks *but not* mortality.

- However, GMIB has a life-related annuity; thus, *longevity risk, which is a non-diversifiable risk* **cannot** *be ignored.*

- Deelstra and Raye (2013) - valuation in a local volatility model; survival rates based on a mortality table. But, mortality need to be stochastic to capture long-run uncertainty (esp. maturities > 10 years).

Stochastic mortality models & dependence of risk factors

- **Evolution of some well-known mortality models**
  – one-parameter model for trends observed in U.S. population data (Lee & Carter, 1992)
  – evaluation of eight stochastic models at advanced ages (Cairns et al., 2009)
  – affine model calibration to different generations of UK population (Luciano & Vigna, 2008)
  – mean-reverting models, with variable target for age-related increase in mortality, outperform non-mean reverting models (Zeddouk & Devolder, 2020).

- **Prevalent assumption!** independence of mortality risk from interest risk.
  – within risk-neutral domain, such assumption is frequently unattainable (Dhaene et al., 2013)
  – mortality influenced the economy *subsequently impacting* interest rate (Miltersen and Persson, 2005 & Liu et al., 2014)
  – *implications of dependence* between mortality and interest risks *on insurance prices* (Deelstra et al., 2016).

- There is merit to a mathematical framework for *dependence structure* between financial and mortality risks.

Key contributions of this research

(i) Further developments in constructing equivalent martingale measures [some parallels to Dahl and Miller (2006)].

(ii) GMIB rider with correlated stochastic interest and mortality rates extending Bauer et al. (2008) and Marshall et al. (2010).

(iii) With endowment-risk-adjusted measure, analytical solution for GMIB is derived for 2 Benefit Base function scenarios.

(iv) Remarkably accurate GMIB prices obtained with significantly reduced computation time vis-á-vis results from standard Monte Carlo simulation.

(v) Comprehensive assessment of various risk factors' impact on GMIB value.

(vi) Flexibility of modelling framework along with change of numéraire approach for other types of guarantee riders - practical utility in insurance industry.

Uncertainty risks affecting GMIB

Three uncertainty risks:

- interest rate r_t (e.g., $\mathrm{d}r_t = a^* (\theta^*(t) - r_t)\,\mathrm{d}t + \sigma_1^* \mathrm{d}X_t$.)
- mortality intensity μ_t: (modelled similarly by some stoch process with starred parameters)
- investment fund S_t: (also modelled by a certain stoch process with starred parameters)

They are defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, where
$\{\mathcal{F}_t\}$ is the joint filtration generated by r_t, μ_t, and S_t,
with P being an objective probability measure.

Note:*Procedure for effectuating change of measure from P to a risk-neutral measure Q was developed based on the methodology of Dahl and Moller (2006), and hopefully could be shown if time permits (end of this slide presentation).*

The interest rate model

Under Q, the process r_t follows the Hull-White model

$$\mathrm{d}r_t = a(\theta(t) - r_t)\mathrm{d}t + \sigma_1 \mathrm{d}X_t. \tag{1}$$

In (1), $a > 0$ and $\sigma_1 > 0$, $\theta(t)$ is deterministic describing initial interest rate's term structure, and X_t is a standard Brownian motion (BM).

The price $B(t, T)$ of a T-maturity zero-coupon bond at time $t < T$ is

$$B(t, T) = \mathbb{E}^Q \left[e^{-\int_t^T r_u \mathrm{d}u} \middle| \mathcal{F}_t \right] = e^{-A(t,T)r_t + D(t,T)}, \tag{2}$$

where

$$A(t, T) = \frac{1 - e^{-a(T-t)}}{a} \tag{3}$$

and

$$\begin{aligned} D(t, T) = &- \int_t^T \left(1 - e^{-a(T-u)}\right) \theta(u)\mathrm{d}u \\ &+ \frac{\sigma_1^2}{4a^3} \left[2a(T-t) - 3 + 4e^{-a(T-t)} - e^{-2a(T-t)}\right]. \end{aligned} \tag{4}$$

The mortality rate model

- $\mu_{x,t}$:= time-t force of mortality of an individual aged x at time 0.
- $\mu_{x,t}$ is specified by

$$\mathrm{d}\mu_{x,t} = c\left(\xi(t) - \mu_{x,t}\right)\mathrm{d}t + \sigma_2 \mathrm{d}Y_t, \tag{5}$$

  where $c > 0$ and $\sigma_2 > 0$, $\xi(t)$ is deterministic, and Y_t is a standard BM.
- Parameters in (5) could be set such that probability for $\mu_{x,t}$ to ever become negative is minimised.
- X_t and Y_t are correlated, i.e. $\mathrm{d}X_t \mathrm{d}Y_t = \rho \mathrm{d}t$.
- Following Zedouk and Devolder (2020), $\xi(t)$ conforms to Gompertz function, i.e., $\xi(t) = pe^{ht}$. Mortality intensity exponentially grows with advancing age; p = baseline mortality at age x and h = senescent component. *To streamline notation, the age index x is omitted.*
- Model justification: See Luciano & Vigna (2008) - insignificant prob of neg rates; and Costabile et al. (2025) - truncated at 0 when rates are neg.

The investment fund model

- The investment fund S_t of a VA has geometric BM dynamics:

$$\mathrm{d}S_t = r_t S_t \mathrm{d}t + \sigma_3 S_t \mathrm{d}Z_t, \tag{6}$$

where $\sigma_3 > 0$, and Z_t is a standard BM independent of X_t and Y_t.

- To ensure a consistent correlation matrix for simulation and other financial modelling purposes, the following relation dynamics must be satisfied:

$$\mathrm{d}X_t = \mathrm{d}W_t^1, \quad \mathrm{d}Y_t = \rho \mathrm{d}W_t^1 + \sqrt{1-\rho^2}\mathrm{d}W_t^2, \quad \text{and} \quad \mathrm{d}Z_t = \mathrm{d}W_t^3,$$

where W_t^1, W_t^2 and W_t^3 are independent standard BMs.

Specifications and features of GMIB product

- (i) VA with maturity T, initial premium P_0 invested in S_t. (ii) Policyholder's fund F_t, held in a separate account, is linked to performance of S_t. (iii) Continuously compounded management charge rate α. So,

$$F_t = F_0 \frac{S_t}{S_0} e^{-\alpha t},$$

where $F_0 = S_0 = P_0$. By Itô's lemma, policyholder's fund F_t satisfies

$$\mathrm{d}F_t = (r_t - \alpha) F_t \mathrm{d}t + \sigma_3 F_t \mathrm{d}W_t^3. \tag{7}$$

- GMIB rider - offers **guaranteed** annuitisation rate g, an annual income amount per unit of a lump sum. At date T, if g is better than prevailing market annuitisation rates, option to utilise GMIB rider is triggered. A predetermined minimum sum of funds, called **benefit base** for a life annuity at rate g.

- GMIB is a survivor benefit, i.e., annuitisation does not occur if policyholder's demise occurs before T.

Specifications: GMIB product (cont'd)

- $BB_T :=$ Benefit Base if policy matures at time T. Two variations are adopted.
- Benefit Base I: growth at a guaranteed rate δ (a roll-up feature); so

$$BB_T = P_0 e^{\delta T}.$$

- Benefit Base II: incorporates discrete lookback feature (i.e., a step-up guarantee). Let $0 = t_1 < t_2 < \cdots < t_m = T$ be some pre-selected policy anniversaries. Benefit Base II is of the form

$$BB_T = \max\left(P_0 e^{\delta T}, F_{t_1}, F_{t_1}, \ldots, F_{t_m}\right).$$

Locks in gains when investment returns are strong during accumulation phase.

Specifications: GMIB product (cont'd)

- $M(t, T^*)$: time t value of \$1 pure endowment payable at T^*. Thus,

$$M(t, T^*) = \mathbb{E}^Q \left[e^{-\int_t^{T^*} r_u \mathrm{d}u} e^{-\int_t^{T^*} \mu_u \mathrm{d}u} \middle| \mathcal{F}_t \right].$$

- n-year term life annuity is the sum of pure endowments; that is,

$$\ddot{a}_{x+T:\overline{n}|} = \sum_{k=0}^{n-1} M(T, T+k).$$

- Assume no lapse risk for now. GMIB payoff at T, conditional on policyholder's survival is

$$\max(BB_T g \ddot{a}_{x+T:\overline{n}|} - F_T, 0).$$

- Consequently, GMIB's fair value at time t is

$$P_{\text{GMIB}}(t) = \mathbb{E}^Q \left[e^{-\int_t^T r_u \mathrm{d}u} e^{-\int_t^T \mu_u \mathrm{d}u} \max(BB_T g \ddot{a}_{x+T:\overline{n}|} - F_T, 0) \middle| \mathcal{F}_t \right]. \quad (8)$$

The GMIB analytical price representation

- We employ the change of measure technique to carry out the pricing evaluation of the GMIB.
- This is accomplished by introducing the forward measure to obtain a closed-form solution for the pure endowment $M(t, T)$.

The forward measure

- Bond price $B(t, T^*)$ is a numéraire associated with forward measure $\widetilde{Q}$ equivalent to risk-neutral measure Q via

$$\left.\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}Q}\right|_{\mathcal{F}_{T^*}} = \Lambda^1_{T^*} := \frac{e^{-\int_0^{T^*} r_u \mathrm{d}u} B(T^*, T^*)}{B(0, T^*)}.$$

- By Bayes' rule for conditional expectation,

$$\begin{aligned} M(t, T^*) &= \mathbb{E}^Q\left[e^{-\int_t^{T^*} r_u \mathrm{d}u} e^{-\int_t^{T^*} \mu_u \mathrm{d}u} \middle| \mathcal{F}_t \right] \\ &= B(t, T^*) \mathbb{E}^{\widetilde{Q}}\left[e^{-\int_t^{T^*} \mu_u \mathrm{d}u} \middle| \mathcal{F}_t \right] = B(t, T^*) S(t, T^*). \end{aligned} \tag{9}$$

- Our calculations show

$$\mathrm{d}\Lambda^1_t = -\sigma_1 A(t, T^*) \Lambda^1_t \mathrm{d}W^1_t.$$

By Girsanov's Theorem, $\widetilde{W}^1_t$ and $\widetilde{W}^2_t$ are standard BMs under $\widetilde{Q}$,

$$\mathrm{d}\widetilde{W}^1_t = \mathrm{d}W^1_t + \sigma_1 A(t, T^*) \mathrm{d}t \quad \text{and} \quad \mathrm{d}\widetilde{W}^2_t = \mathrm{d}W^2_t.$$

The forward measure (cont'd)

- $\widetilde{Q}$ dynamics of r_t and μ_t:

$$\begin{aligned} \mathrm{d}r_t &= [a\theta(t) - \sigma_1^2 A(t, T^*) - ar_t]\mathrm{d}t + \sigma_1 \mathrm{d}\widetilde{W}_t^1, \\ \mathrm{d}\mu_t &= [cpe^{ht} - \rho\sigma_1\sigma_2 A(t, T^*) - c\mu_t]\mathrm{d}t + \sigma_2 \left(\rho \mathrm{d}\widetilde{W}_t^1 + \sqrt{1-\rho^2}\mathrm{d}\widetilde{W}_t^2\right). \end{aligned} \tag{10}$$

- Eq. (10) is solved by variation of constants; note $\int_t^{T^*} \mu_u \mathrm{d}u$ conditional on $\mathcal{F}_t$ is normally distributed with deterministic mean m_1 and variance v_1.

- We obtain

$$S(t, T^*) = \mathbb{E}^{\widetilde{Q}}\left[e^{-\int_t^{T^*} \mu_u \mathrm{d}u} \middle| \mathcal{F}_t\right] = e^{-m_1 + \frac{1}{2}v_1} = e^{-\mu_t \widetilde{G}(t,T^*) + \widetilde{H}(t,T^*)}, \tag{11}$$

where $\widetilde{G}(t, T^*)$ and $\widetilde{H}(t, T^*)$ are deterministic.

- Therefore, pure endowment has closed-form expression

$$M(t, T^*) = e^{-\left(A(t,T^*)r_t + \widetilde{G}(t,T^*)\mu_t\right) + D(t,T^*) + \widetilde{H}(t,T^*)}. \tag{12}$$

- Consequently, term annuity factor is

$$\ddot{a}_{x+T:\overline{n}|} = \sum_{k=0}^{n-1} M(T, T+k) = \sum_{k=0}^{n-1} e^{-\left(A(T,T+k)r_T + \widetilde{G}(T,T+k)\mu_T\right) + D(T,T+k) + \widetilde{H}(T,T+k)}. \tag{13}$$

The endowment-risk-adjusted measure

- Consider $M(t,T)$ as numéraire (linked to endowment-risk-adjusted measure $\widehat{Q}$) defined via
$$\left.\frac{\mathrm{d}\widehat{Q}}{\mathrm{d}Q}\right|_{\mathcal{F}_T} = \Lambda_T^2 := \frac{e^{-\int_0^T r_u \mathrm{d}u} e^{-\int_0^T \mu_u \mathrm{d}u} M(T,T)}{M(0,T)}.$$
- By Bayes' rule, equation (8) is
$$\begin{aligned} P_{\mathrm{GMIB}}(t) &= \mathbb{E}^Q\left[e^{-\int_t^T r_u \mathrm{d}u} e^{-\int_t^T \mu_u \mathrm{d}u} \max(BB_T g \ddot{a}_{x+T:\overline{n}|} - F_T, 0) \middle| \mathcal{F}_t \right] \\ &= M(t,T)\mathbb{E}^{\widehat{Q}}\left[\max(BB_T g \ddot{a}_{x+T:\overline{n}|} - F_T, 0) | \mathcal{F}_t\right]. \end{aligned} \tag{14}$$
- For $BB_T = P_0 e^{\delta t}$, the expectation in (14) relies solely of r_T, μ_T and F_T.
- For Benefit Base II, $BB_T = \max(P_0 e^{\delta t}, F_{t_1}, F_{t_2}, \cdots, F_{t_m})$, the expectation in (14) depends on r_T, μ_T, F_{t_1}, $F_{t_2}, \cdots$, F_{t_m}, noting that $t_m = T$.
- Understanding of $\widehat{Q}$-dynamics governing r_t, μ_t and F_t is essential.

The endowment-risk-adjusted measure (cont'd)

- Write $\Lambda_t^2 := \dfrac{Y_t M_t}{M(0,T)}$, where

$$Y_t = e^{-\int_0^t r_u \mathrm{d}u} B(t,T) \quad \text{and} \quad M_t = e^{-\int_0^t \mu_u \mathrm{d}u} S(t,T).$$

- Using Itô's lemma,

$$\begin{aligned}\mathrm{d}Y_t &= -\sigma_1 A(t,T) Y_t \mathrm{d}W_t^1, \quad \text{and} \\ \mathrm{d}M_t &= -\rho\sigma_1\sigma_2 A(t,T)\widetilde{G}(t,T) M_t \mathrm{d}t - \rho\sigma_2 \widetilde{G}(t,T) M_t \mathrm{d}W_t^1 - \sqrt{1-\rho^2}\sigma_2 \widetilde{G}(t,T) M_t \mathrm{d}W_t^2.\end{aligned}$$

- Dynamics of Λ_t^2:

$$\begin{aligned}\mathrm{d}\Lambda_t^2 = \mathrm{d}\left(\frac{Y_t M_t}{M(0,T)}\right) = -\Lambda_t^2 \Bigg[&\left(\sigma_1 A(t,T) + \rho\sigma_2 \widetilde{G}(t,T)\right) \mathrm{d}W_t^1 \\ &+ \sqrt{1-\rho^2}\sigma_2 \widetilde{G}(t,T)\mathrm{d}W_t^2 \Bigg]. \end{aligned} \tag{15}$$

- Girsanov's Theorem justifies that $\widehat{W}_t^1$, $\widehat{W}_t^2$ and $\widehat{W}_t^3$ are standard $\widehat{Q}$-BMs:

$$\mathrm{d}\widehat{W}_t^1 = \mathrm{d}W_t^1 + \left(\sigma_1 A(t,T) + \rho\sigma_2 \widetilde{G}(t,T)\right)\mathrm{d}t, \quad \mathrm{d}\widehat{W}_t^2 = \mathrm{d}W_t^2 + \sqrt{1-\rho^2}\sigma_2 \widetilde{G}(t,T)\mathrm{d}t,$$

and $\mathrm{d}\widehat{W}_t^3 = \mathrm{d}W_t^3$.

Valuation formula: preliminaries

- Respective $\widehat{Q}$-dynamics of r_t, μ_t and F_t are given by

$$\mathrm{d}r_t = \left[a\theta(t) - \sigma_1^2 A(t,T) - \rho\sigma_1\sigma_2\widetilde{G}(t,T) - ar_t\right]\mathrm{d}t + \sigma_1 \mathrm{d}\widehat{W}_t^1, \tag{16}$$

$$\mathrm{d}\mu_t = \left[cpe^{ht} - \rho\sigma_1\sigma_2 A(t,T) - \sigma_2^2\widetilde{G}(t,T) - c\mu_t\right]\mathrm{d}t + \rho\sigma_2\mathrm{d}\widehat{W}_t^1 + \sqrt{1-\rho^2}\sigma_2\mathrm{d}\widehat{W}_t^2, \tag{17}$$

$$\mathrm{d}F_t = (r_t - \alpha)F_t dt + \sigma_3 F_t \mathrm{d}\widehat{W}_t^3. \tag{18}$$

- RVs r, μ and Y are normally dstributed with deterministic moments for $u > t$: $m_r(t,u)$, ${\sigma_r}^2(t,u)$, $m_\mu(t,u)$, ${\sigma_\mu}^2(t,u)$, $m_Y(t,u)$, and ${\sigma_Y}^2(t,u)$.

- From their dynamics and conditional on $\mathcal{F}_t$, $\{r_T,\ \mu_T,\ Y_{t,\ T}\}$ follows a trivariate normal distribution.

- Hence, the pertinent expectation involving $\{r_T,\ \mu_T,\ Y_{t,\ T}\}$, can be computed efficiently.

GMIB under Benefit Base I

Theorem 1

The GMIB value under Benefit Base I at time $t \leq T$ is

$$P^{(I)}_{GMIB}(t) = M(t,T) \times$$
$$\mathbb{E}^{\widehat{Q}}\left[\max\left(gP_0 e^{\delta T}\sum_{k=0}^{n-1} e^{-\left(A(T,T+k)r_T+\widetilde{G}(T,T+k)\mu_T\right)+D(T,T+k)+\widetilde{H}(T,T+k)} - F_t e^{Y_{t,\ T}}, 0\right)\middle|\mathcal{F}_t\right], \quad (19)$$

where the pure endowment $M(t,T)$ is defined in equation (12).

Moreover, the conditional distribution of $\{r_T,\ \mu_T,\ Y_{t,\ T}\}$ given $\mathcal{F}_t$ is a trivariate normal with (deterministic) parameters:
$\mathbb{E}^{\widehat{Q}}[r_T|\mathcal{F}_t]$, $\mathbb{E}^{\widehat{Q}}[\mu_T|\mathcal{F}_t]$, $\mathbb{E}^{\widehat{Q}}[Y_{t,\ T}|\mathcal{F}_t]$, $\mathbb{V}ar^{\widehat{Q}}[r_T|\mathcal{F}_t]$,
$\mathbb{V}ar^{\widehat{Q}}[\mu_T|\mathcal{F}_t]$, $\mathbb{V}ar^{\widehat{Q}}[Y_{t,\ T}|\mathcal{F}_t]$, $\mathbb{C}ov^{\widehat{Q}}[r_T,\mu_T|\mathcal{F}_t]$,
$\mathbb{C}ov^{\widehat{Q}}[r_T, Y_{t,\ T}|\mathcal{F}_t]$, *and* $\mathbb{C}ov^{\widehat{Q}}[\mu_T, Y_{t,\ T}|\mathcal{F}_t]$.

GMIB under Benefit Base II

- **Recall:** $BB_T = \max(P_0 e^{\delta t}, F_{t_1}, F_{t_2}, \cdots, F_{t_m})$. Equation (14) is dependent only on r_T, μ_T, and F_{t_i}. Define the index set $I_t = \{j : j = 1, 2, \cdots, m, t_j > t\}$ and $\overline{I_t} = \{j : j = 1, 2, \cdots, m, t_j \leq t\}$. So, $F_{t_i} = F_t e^{Y_{t,\ t_i}}$, $i \in I_t$.

Theorem 2

The GMIB value under Benefit Base II at time $t \leq T$ is

$$P^{(II)}_{GMIB}(t) = M(t, T) \times \mathbb{E}^{\widehat{Q}}\left[\max\left\{ g \max\left(P_0 e^{\delta T}, \max_{i \in \overline{I_t}} F_{t_i}, \max_{i \in I_t} F_t e^{Y_{t,\ t_i}} \right) \times \sum_{k=0}^{n-1} e^{-\left(A(T,T+k) r_T + \widetilde{G}(T,T+k)\mu_T\right) + D(T,T+k) + \widetilde{H}(T,T+k)} - F_t e^{Y_{t,\ T}}, 0 \right\} \middle| \mathcal{F}_t \right], \tag{20}$$

where $\{r_T,\ \mu_T,\ Y_{t,\ t_i}\}$, for $i \in I_t$ conditional on $\mathcal{F}_t$ has a multivariate normal distribution with (deterministic) parameters:
$\mathbb{E}^{\widehat{Q}}[r_T|\mathcal{F}_t]$, $\mathbb{E}^{\widehat{Q}}[\mu_T|\mathcal{F}_t]$, $\mathbb{E}^{\widehat{Q}}[Y_{t,\ t_i}|\mathcal{F}_t]$, $\mathbb{V}ar^{\widehat{Q}}[r_T|\mathcal{F}_t]$, $\mathbb{V}ar^{\widehat{Q}}[\mu_T|\mathcal{F}_t]$, $\mathbb{V}ar^{\widehat{Q}}[Y_{t,\ t_i}|\mathcal{F}_t]$,
$\mathbb{C}ov^{\widehat{Q}}[r_T, \mu_T|\mathcal{F}_t]$, $\mathbb{C}ov^{\widehat{Q}}[r_T, Y_{t,\ t_i}|\mathcal{F}_t]$, *and* $\mathbb{C}ov^{\widehat{Q}}\left[Y_{t,\ t_i}, Y_{t,\ t_j}\middle|\mathcal{F}_t\right]$.

The benchmark: Standard MC (Glasserman, 2004 & Kroese et al., 2013)

1. Generate j sequences of independent standard normals $\{\varepsilon^{1,j}_{u_i}, \varepsilon^{2,j}_{u_i}, \varepsilon^{3,j}_{u_i}\}$, $i = 1, 2, \ldots, k$, for k sub-intervals in each $j - th$ sequence and $j = 1, 2, \ldots, N$.

2. Generate the j-th-sample path $(j = 1, 2, \ldots, N)$ of r_t, μ_t and F_t according to the Euler-Maruyama discretisation, respecting the correlation structure.

3. The j-th GMIB value is

$$P^j_{\text{GMIB}}(t) = e^{-D^j_r} e^{-D^j_\mu} \max(BB^j_T g \ddot{a}_{x+T:\overline{n|}}{}^j - F^j_T, 0)$$

   for numerically computed discounted factors D^j_r and D^j_μ.

4. Approximate the GMIB value by

$$P_{\text{GMIB}}(t) \approx \frac{1}{N} \sum_{j=1}^{N} P^j_{\text{GMIB}}(t),$$

   and report the standard error.

The proposed approach

- Generate N sequences: trivariate normals $\{r_T^j,\ \mu_T^j,\ Y_{t,\ T}^j\}, j = 1, 2, \dots, N$.

- With $M(t, T)$ having a closed form (12), the j-th GMIB values for Bases I and II based on Theorems 4.1 and 4.2., respectively, are:

$$P_{\text{GMIB}}^{(\text{Base I}),j}(t) = M(t, T)$$
$$\times\, \mathbb{E}^{\widehat{Q}}\left[\max\left(g\ P_0 e^{\delta T}\sum_{s=0}^{n-1} e^{-\left(A(T,T+s)r_T^j+\widetilde{G}(T,T+s)\mu_T^j\right)+D(T,T+s)+\widetilde{H}(T,T+s)} - F_t e^{Y_{t,\ T}^j}, 0\right)\right];$$

$$P_{\text{GMIB}}^{(\text{Base II}),j}(t) = M(t, T) \times \mathbb{E}^{\widehat{Q}}\Bigg[$$
$$\max\left(g \max\left(P_0 e^{\delta T}, \max_{i\in\overline{I_t}} F_{t_i}, \max_{i\in I_t} F_t e^{Y_{t,\ t_i}^j}\right)\sum_{s=0}^{n-1} e^{-\left(A(T,T+s)r_T^j+\widetilde{G}(T,T+s)\mu_T^j\right)+D(T,T+s)+\widetilde{H}(T,T+s)}\right.$$
$$\left. - F_t e^{Y_{t,\ T}^j}, 0\right)\Bigg].$$

- Compute $P_{\text{GMIB}}^{(Base\ \ I\ \ or\ \ II)}(t) \approx \frac{1}{N}\sum_{j=1}^{N} P_{\text{GMIB}}^{(I),j}(t)$ and report the standard error.

Numerical results: setting and assumptions

- $N = 200,000$ sample paths were generated in RStudio.
- Parallel-simulation technique is executed via machine (i7-10700 CPU @ 2.90 GHz, 16 Cores, 64GB Memory).
- Risk factors' parameters and GMIB contract specification are given in next slide.
- The mortality model parameters are based on Zeddouk and Devolder (2020).
- GMIB is based on cohort aged 50 at $t = 0$. Policyholder is assumed to hold contract until $T = 10$ (age 60). Then, they will receive a 20-year term life annuity-due, with annual payments from age 60 to age 79.

Parameter setting assumptions

Table 1: Parameter values

Interest rate model				
$a = 0.15$	$\theta = 0.045$	$\sigma_1 = 0.03$	$r_0 = 0.045$	
Mortality model				
$c = 0.4496$	$p = 0.0091$	$h = 0.0847$	$\sigma_2 = 0.027$	$\mu_0 = 0.0079$
Policyholder's fund				
$\alpha = 0.01$	$\sigma_3 = 0.3$	$F_0 = 1$		
GMIB contract specification				
$T = 10$	$\delta = 0.03$	$g = 0.06$	$n = 20$	$P_0 = 1$
Base Benefit I	$BB_T = P_0 e^{\delta T}$			
Base Benefit II	$BB_T = \max\left(P_0 e^{\delta T}, F_0, F_5, F_T\right)$			

Pricing results

Table 2: GMIB value at time $t = 0$ with **Benefit Base I**

ρ	The MC Benchmark Eq. (8)	Our proposed approach Eq. (19)
-0.9	0.14822 (0.00047)	0.14819 (0.00040)
-0.7	0.15594 (0.00050)	0.15635 (0.00042)
-0.5	0.16482 (0.00055)	0.16490 (0.00044)
-0.3	0.17317 (0.00058)	0.17387 (0.00046)
-0.1	0.18346 (0.00064)	0.18325 (0.00048)
0.0	0.18847 (0.00066)	0.18857 (0.00049)
0.2	0.19886 (0.00072)	0.19865 (0.00051)
0.4	0.20858 (0.00078)	0.20921 (0.00053)
0.6	0.22026 (0.00084)	0.22029 (0.00055)
0.8	0.23200 (0.00090)	0.23191 (0.00058)
0.9	0.23702 (0.00093)	0.23793 (0.00059)
Average computing time	331.68 secs	**0.25 secs**

Pricing results

Table 3: GMIB value at time $t = 0$ with **Benefit Base II**

ρ	The MC Benchmark eq. (8)	Our proposed approach eq. (20)
-0.9	0.16917 (0.00052)	0.16882 (0.00045)
-0.7	0.17855 (0.00056)	0.17836 (0.00047)
-0.5	0.18911 (0.00061)	0.18843 (0.00049)
-0.3	0.19864 (0.00066)	0.19905 (0.00051)
-0.1	0.20954 (0.00071)	0.21025 (0.00054)
0.0	0.21655 (0.00074)	0.21623 (0.00055)
0.2	0.22895 (0.00080)	0.22836 (0.00058)
0.4	0.24156 (0.00087)	0.24116 (0.00060)
0.6	0.25451 (0.00094)	0.25465 (0.00063)
0.8	0.26916 (0.00100)	0.26886 (0.00066)
0.9	0.27682 (0.00105)	0.27624 (0.00068)
Average computing time	333.57 secs	**0.26 secs**

Sensitivity analysis: θ & σ_1 for interest rate

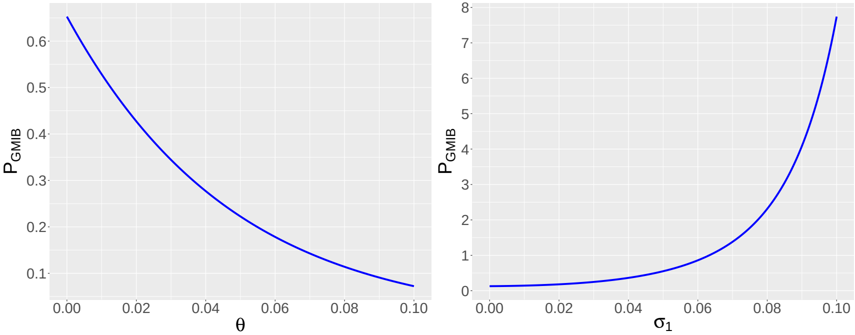

Figure 1: GMIB value at time $t = 0$ as a function of θ and σ_1

Sensitivity analysis: p, h & σ_2 for μ

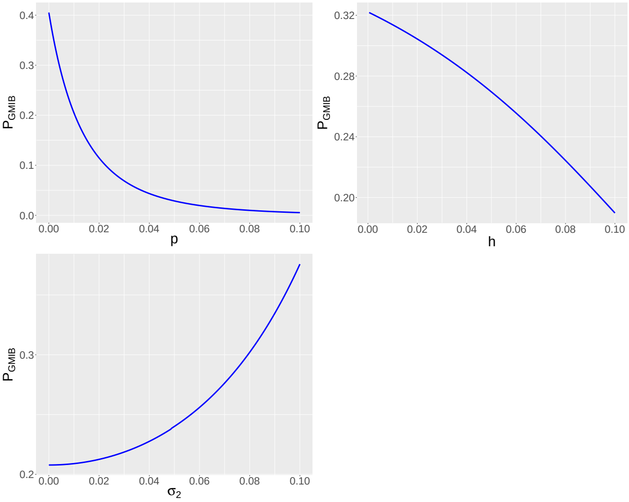

Figure 2: GMIB value at time $t = 0$ as a function of p, h and σ_2

Sensitivity analysis: investment fund volatility σ_3

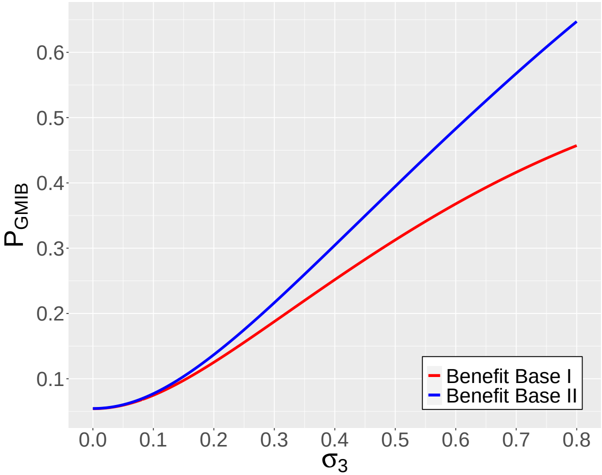

Figure 3: GMIB value at $t = 0$ as a function of σ_3

Sensitivity analysis: roll-up δ and g

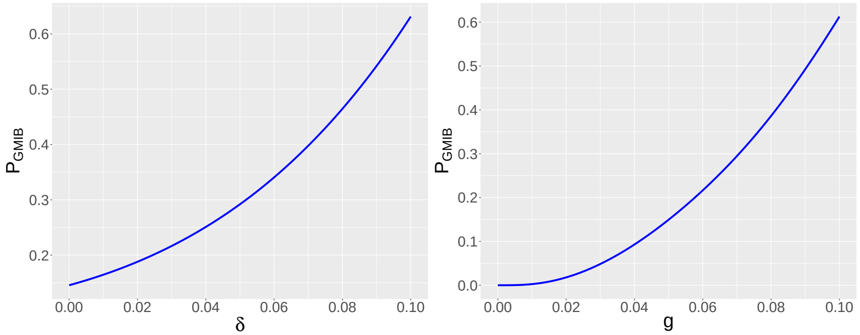

Figure 4: GMIB value at time $t = 0$ as a function of δ and g

Sensitivity analysis: annuity term

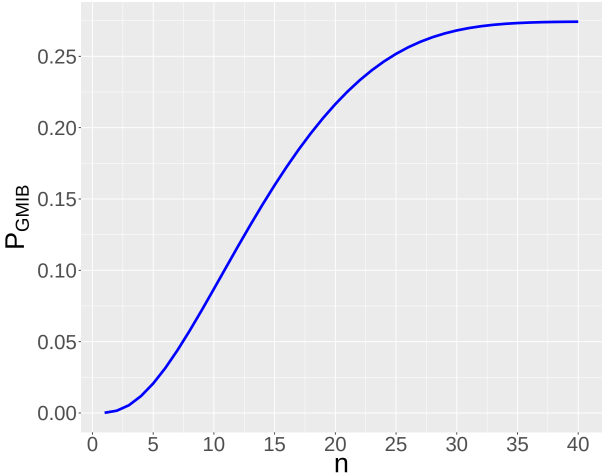

Figure 5: GMIB value at $t = 0$ as a function of the annuity term n

Sensitivity analysis: contract maturity

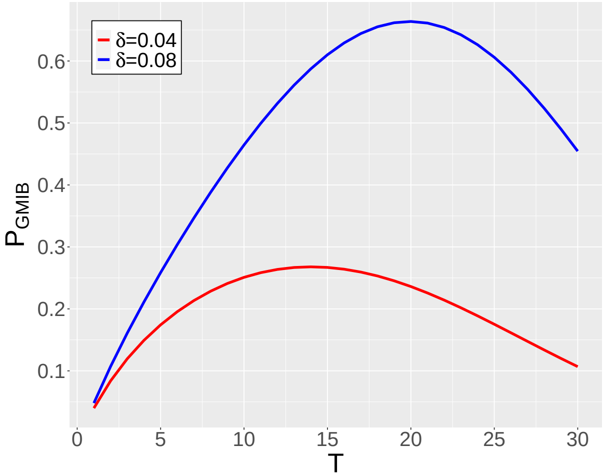

Figure 6: GMIB value at $t = 0$ across various maturities T

Impact of mortality risk

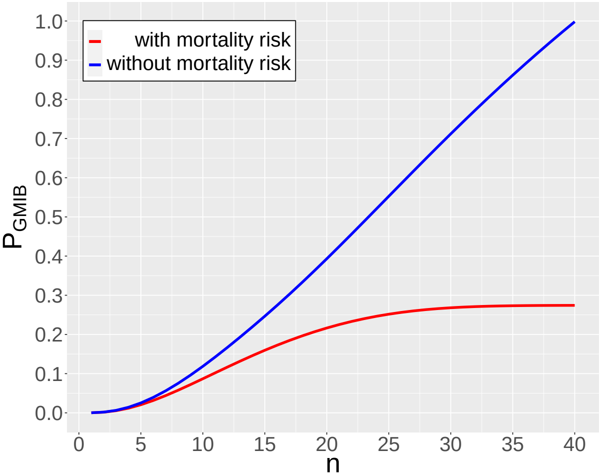

Figure 7: GMIB value with mortality risk versus GMIB value without mortality risk

Simplified approach to examine lapse risk's effect

- $\pi_i :=$ lapsation probability within time period $[i-1, i]$, corresponding to policy year $i = 1, 2, \ldots, 10$.
- Upon lapsation, the right to receive guaranteed annuitisation rate at maturity is forfeited, whilst insurer retains fees earned from providing GMIB rider.
- Fair value of lapse-risk-adjusted GMIB at time $t = 0$ is:

$$P^*_{\text{GMIB}}(0) = \pi_1 \cdot 0 + \sum_{i=1}^{T-1} \prod_{j=1}^{i} (1-\pi_j)\pi_{i+1} \cdot 0 + \prod_{j=1}^{T}(1-\pi_j) \cdot \mathbb{E}^Q \left[e^{-\int_t^T r_u \mathrm{d}u} e^{-\int_t^T \mu_u \mathrm{d}u} \max(BB_T g \ddot{a}_{x+T:\overline{n}|} - F_T, 0) \middle| \mathcal{F}_t \right] = \prod_{j=1}^{T}(1-\pi_j) P_{\text{GMIB}}(0),$$

where $P_{\text{GMIB}}(0)$ is GMIB value at $t = 0$ without lapse risk.

Impact of lapse risk

Table 4: Lapse-risk-adjusted GMIB value

π_i	$\pi_i \equiv 2\% \quad \forall\ i$	$\pi_i \equiv 5\% \quad \forall\ i$
$\frac{P^*_{\text{GMIB}}(0)}{P_{\text{GMIB}}(0)}$	81.71 %	59.87 %
π_i	$\pi_i = 5\%,\ i = 1, \ldots, 5;\ \pi_i = 2\%,\ i = 6, \ldots, 10$	
$\frac{P^*_{\text{GMIB}}(0)}{P_{\text{GMIB}}(0)}$	69.94 %	

- Magnitude of GMIB-value decline depends on lapse probability assumptions.
- In each scenario, GMIB value is sensitive to lapse risk's fluctuations.
- Robust methodologies are needed for assessing and managing lapse risk for reserve/capital-level adequacy.

Processes under measure P

- This connects outcomes obtained under Q framework and those under P setting, where empirical data are utilised for model calibration.

- Assume P-dynamics of r_t, μ_t and S_t are given by

$$
\begin{aligned}
\mathrm{d}r_t &= a^*(\theta_1^*(t) - r_t)\mathrm{d}t + \sigma_1 \mathrm{d}W_t^{1,P}, \\
\mathrm{d}\mu_t &= c^*(\xi_1^*(t) - \mu_t)\mathrm{d}t + \sigma_2 \left(\rho \mathrm{d}W_t^{1,P} + \sqrt{1-\rho^2}\mathrm{d}W_t^{2,P}\right), \\
\mathrm{d}S_t &= u_t S_t \mathrm{d}t + \sigma_3 S_t \mathrm{d}W_t^{3,P},
\end{aligned}
$$

where a^*, σ_1, c^*, σ_2 and σ_3 are positive constants, $W_t^{1,P}$, $W_t^{2,P}$ and $W_t^{3,P}$ are independent standard P-BMs.

Girsanov density: Link from P to Q

- Following Dahl & Moller (2006), we construct a likelihood Λ_t^P:

$$\mathrm{d}\Lambda_t = -\Lambda_t \left[f_r(t)\mathrm{d}W_t^{1,P} + f_\mu(t)\mathrm{d}W_t^{2,P} + f_S(t)\mathrm{d}W_t^{3,P} \right], \quad \Lambda_0 = 1,$$

$$f_r(t) = \frac{c_1\theta_2^*(t) + c_2 r_t}{\sigma_1}, \quad f_S(t) = \frac{u_t - r_t}{\sigma_2}$$

$$f_\mu(t) = \frac{c_3\sigma_1\xi_2^*(t) - c_1\sigma_2\rho\theta_2^*(t) - c_2\sigma_2\rho r_t + c_4\sigma_1\mu_t}{\sigma_1\sigma_2\sqrt{1-\rho^2}},$$

  where c_1, c_2, c_3 and c_4 are constants with $c_1 < 0$, $c_2 > -a^*$, $c_3 < 0$ and $c_4 > -c^*$.
- Consequently, $\mathrm{d}W_t^{1,Q} = \mathrm{d}W_t^{1,P} + f_r(t)\mathrm{d}t$, $\mathrm{d}W_t^{2,Q} = \mathrm{d}W_t^{2,P} + f_\mu(t)\mathrm{d}t$, $\mathrm{d}W_t^{3,Q} = \mathrm{d}W_t^{3,P} + f_S(t)\mathrm{d}t$.
- Also, $a = a^* + c_2$, $\theta(t) = \frac{a^*\theta_1^*(t) - c_1\theta_2^*(t)}{a^* + c_2}$, $c = c^* + c_4$, $\xi(t) = \frac{c^*\xi_1^*(t) - c_3\xi_2^*(t)}{c^* + c_4}$.

Concluding remarks

- New stochastic modelling framework GMIB valuation: dependence structure between interest & mortality rates and examination of lapse risk effect.
- Change of numéraire technique to obtain quasi closed-form GMIB valuation formula.
- Numerical experiments: Proposed approach versus MC simulation (benchmark).
- Superior accuracy and efficiency of our proposed approach over benchmark.
- Sensitivity analyses: impact of individual parameters on GMIB value $\longrightarrow$ insights to insurers & regulators.

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